

# Diophantine Approximation of non-algebraic points on varieties II: Explicit estimates for Arithmetic Hilbert Functions

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## 1 Introduction

Let  $k$  be a number field with ring of integers  $\mathcal{O}_k$ ,  $\mathcal{X}$  a projective flat scheme of relative dimension  $t$  over  $\mathrm{Spec} \mathcal{O}_k$ , which in this paper is referred to by the term of a  $t$ -dimensional arithmetic variety, and  $\bar{\mathcal{L}}$  a positive metrized line bundle on  $\mathcal{X}$ . If for  $D \in \mathbb{N}$  one defines  $H_X(D)$  as the dimension of the vector space of global sections of  $\mathcal{L}^{\otimes D}$  on  $X = \mathcal{X} \times_{\mathrm{Spec} \mathcal{O}_k} \mathrm{Spec} k$ , and  $\hat{\mathcal{H}}_{\mathcal{X}}(D)$  as the arithmetic degree of the arithmetic bundle of global sections of  $\bar{\mathcal{L}}^{\otimes D}$  over  $\mathcal{X}$ , there are the well known algebraic and arithmetic Hilbert-Samuel formulas

$$H_X(D) = \deg_{\mathcal{L}} X \frac{D^t}{t!} + O(D^{t-1}),$$

$$\hat{\mathcal{H}}_{\mathcal{X}}(D) = h_{\bar{\mathcal{L}}}(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} + O(D^t \log D).$$

One can define a third kind of Hilbert function: Let  $\sigma : k \rightarrow \mathbb{C}$  be some embedding, and  $\theta \in X(\mathbb{C}_\sigma)$  a generic point, i. e. a point whose algebraic closure over the algebraic closure  $\bar{k}$  of  $k$  is all of  $X$ , assume further that  $X(\mathbb{C}_\sigma)$  is endowed with a Kähler metric such that the Kähler form coincides with the chern form  $\bar{c}_1(\bar{L})$ , and define

$$\Gamma(D, H) := \{f \in \Gamma(\mathcal{X}, \mathcal{L}^{\otimes D}) \mid \log |f|_{L^2(X)_\mathbb{C}} \leq H\},$$

and

$$\bar{H}_{\mathcal{X}, \theta}(D, H) := - \min_{f \in \Gamma(D, H)} \log |f_\theta|. \quad (1)$$

We always assume that  $H$  sufficiently big compared with  $D$ . For  $k = \mathbb{Z}$ , the Theorem of Minkowski together with the algebraic and arithmetic Hilbert-Samuel formulas implies that for sufficiently big  $D$ ,

$$\bar{H}_{\mathcal{X}, \theta}(D, H) \geq h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} + H \deg X \frac{D^t}{t!} + O(D^t \log(DH)). \quad (2)$$

PROOF Let  $D \in \mathbb{N}$ , and  $S_\theta$  be the stalk of  $\mathcal{L}^{\otimes D}$  at  $\theta$ . If  $f \in \Gamma(D, H)_\mathbb{C}$  is a vector of norm one, orthogonal to the kernel of the evaluation map

$$\varphi_D : \Gamma(D, H) \rightarrow S_\theta, \quad f \mapsto f_\theta,$$

and

$$c := \frac{|f|_{L^2(X)}}{|f_\theta|},$$

then, for sufficiently big  $n$ , the vector  $f^{\otimes n}$  is orthogonal to the kernel of the evaluation

$$\varphi_{nD} : \Gamma(nD, H) \rightarrow S_\theta,$$

and

$$\log |f_{L^2(X)}^{\otimes n}| = \log |f_\theta^{\otimes n}| + n \log c.$$

Let now  $\epsilon > 0$  be arbitrary,

$$\log \epsilon_{D, H} = (1 + \epsilon) \left( -h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} - H \deg X \frac{D^t}{t!} \right),$$

and  $I_{\epsilon_{nD, H}} \times K_H$  the product of the interval  $I_{\epsilon_{nD, H}}$  of diameter  $2\epsilon_{nD, H}$  in the direction of  $f^{\otimes n}$  and the cube with width  $e^H$  in the kernel of the evaluation map  $\varphi_{nD}$ , that is orthogonal to  $f^{\otimes n}$ . By the Theorem of Minkowski, and the two Hilbert-Samuel formulas, for sufficiently big  $n$ ,  $I_{\epsilon_{nD, H}} \times K_H$  contains an element  $g \in \Gamma(\mathcal{X}, \mathcal{L}^{nD})$ . Clearly,  $\log |g| \leq H + t \log(nD) - \log(t!)$ , and therefore  $g$  belongs to  $\Gamma(nD, H)$ . Further, the projection of  $g$  to the line generated by  $f^{\otimes n}$  is  $g = af^{\otimes n}$  with  $\log a \leq \epsilon_{nD, H}$ , and hence,

$$\log |g_\theta| = \log |af_\theta^{\otimes n}| = \log |af^{\otimes n}|_{L^2(X)} - n \log c \leq$$

$$(1 + \epsilon) \left( -h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} - H \deg X \frac{D^t}{t!} \right).$$

Since  $\epsilon$  was chosen arbitrary, the claim follows.

The just proved lower bound (2) is equivalent to arithmetic ampleness as stated in [SABK], ch. 8, Theorem 2:

PROOF The cited Theorem states that

$$\#\Gamma(D, H - \log 2) \geq h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} + \deg X (H - \log 2) \frac{D^t}{t!}.$$

Since for each  $f \in \Gamma(D, H)$ ,

$$\log |f|_\theta \leq \log |f|_{L^2(X)} + D \log c \leq H + D \log c,$$

there are vectors  $f, g \in \Gamma(D, H)$  such that

$$\begin{aligned} \log ||f|_\theta - |g|_\theta| &\leq -h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} - \deg X (H - \log 2) \frac{D^t}{t!} + H + D \log c \rightarrow \\ &\quad -h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} - \deg X H \frac{D^t}{t!}. \end{aligned}$$

Thus  $|f - g|_{L^2(X)} \leq H$ , i. e.  $f - g \in \Gamma(D, H)$ , and

$$\log |f - g|_\theta \leq \log ||f|_\theta - |g|_\theta| \leq -h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} - \deg X H \frac{D^t}{t!},$$

which is formula 2.

On the other hand if

$$\#\Gamma(D, H - \log 2) < h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} + \deg X (H - \log 2) \frac{D^t}{t!},$$

then for a generic point  $\theta \in X(\mathbb{C}_\sigma)$ , there is an  $f \in \Gamma(D, H - \log 2)$  such that for every  $g \in \Gamma(D, H)$ ,

$$\log ||f|_\theta - |g|_\theta| > -h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} - \deg X (H - \log 2) \frac{D^t}{t!}.$$

Because the set  $\{f + g | g \in \Gamma(D, H)\}$  contains the set  $\Gamma(D, H - \log 2)$ , this implies that there is no  $h \in \Gamma(D, H - \log 2)$  with

$$\log |h|_\theta < -h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} - \deg X (H - \log 2) \frac{D^t}{t!},$$

and thus

$$\bar{H}_{\mathcal{X}, \theta}(D, H) < h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} + H \deg X \frac{D^t}{t!} + O(D^t \log(DH)).$$

**1.1 Conjecture** *The inequality (2) is an equality for almost all generic  $\theta \in X(\mathbb{C})$ .*

Of course, the term almost all is to be taken in the sense of Lebesgue measure. For  $\mathcal{X} = \mathbb{P}^1$  the projective line (or rather  $\mathcal{X} = \mathbb{A}^1$  the affine line with the corresponding definition of the height of a polynomial), the conjecture is just Sprindjuk's Theorem that almost every transcendental number is an  $S$ -number of order 1 in the sense of Mahler classification ([Ba], Theorem 9.1).

It will be the objective of [Ma3] to proof that there exists a positive constant  $c$  only depending on  $t$  such that

$$\bar{H}_{\mathcal{X},\theta}(D, H) \leq c \left( h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} + H \deg X \frac{D^t}{t!} \right) \quad (3)$$

for almost all generic  $\theta \in X(\mathbb{C})$ , and sufficiently big  $D$ .

By [Ma1], Theorem .2 instead of  $\Gamma(D, H)$  and  $\log |f_\theta|$  one could also use the height  $h(\operatorname{div} f)$ , and the algebraic distance  $D(\theta, \operatorname{div} f)$  in the definition of the transcendental Hilbert function (1). In this perspective, consider the following definitions

$$\Gamma_s(D, H) := \{\mathcal{Y} \in Z_{eff}^s(\mathcal{X}) \mid \deg Y \leq D^s, \quad h(\mathcal{Y}) \leq H D^{s-1}\},$$

where  $Z_{eff}^s(\mathcal{X})$  denotes the semigroup of effective cycles of codimension  $s$  in  $\mathcal{X}$ , and

$$\bar{H}_{\mathcal{X}}^s(D, H) = - \left( \min_{\mathcal{Y} \in \Gamma_s(D, H)} D(Y, \theta) \right),$$

with  $D(Y, \theta)$  the algebraic distance of  $Y$  to  $\theta$  (See [Ma1], p.21 for the definition). It is one of the objectives of [Ma2] to proof

**1.2 Theorem** *There exist positive numbers  $b_1, \dots, b_t$  such that for every generic point  $\theta$ ,*

$$\bar{H}_{\mathcal{X},\theta}^s(D, H) \geq b_s \left( h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} + H \deg X \frac{D^t}{t!} \right).$$

*for all  $s = 1, \dots, t$ , and sufficiently big  $D$ .*

This paper will further contain the stronger estimate

**1.3 Theorem** *There are positive numbers  $\bar{b}_1, \dots, \bar{b}_t$  such that for every generic point  $\theta$ , and for every  $s = 1, \dots, t$  there are cycles  $\mathcal{Y} \in \Gamma_s(D, H)$  such that*

$$\log |Y, \theta| \leq -\bar{b}_t \left( h(\mathcal{X}) \frac{D^{t+1}}{(t+1)!} + H \deg X \frac{D^t}{t!} \right),$$

*where  $|Y, \theta|$  is the distance of  $\theta$  to  $Y$  with respect to the Kähler metric.*

These results should also relate to the analogon to arithmetic ampleness for subvarieties of higher codimension, i. e. a lower bound on the number of points, or subvarieties of arbitrary dimension of bounded height and degree; an upper estimate for this number being comparatively easy to prove.

Modulo a constant, he reversed inequalities of Theorems 1.2 and 1.3 for almost all generic  $\theta$  will also be proved in [Ma3].

The proof of Theorem 1.2 for  $s = 1$  has been given above. The fundamental strategy for proving Theorem 1.2 for cycles of higher codimension is to find effective cycles  $\mathcal{X}_1, \dots, \mathcal{X}_s$  of codimension one that intersect properly and have small algebraic distance to  $\theta$  just as above, and use the metric Bézout Theorem from [Ma1] to prove that their intersection still has small algebraic distance to  $\theta$ . This approach, however, faces the following problem: If one has found  $\mathcal{X}_1$  of bounded height and degree as shown above, in order to find  $\mathcal{X}_2$  that intersects  $\mathcal{X}_1$  properly, one has to use the algebraic and arithmetic Hilbert-Samuel formulas for  $\mathcal{X}_1$ . As these formulas only describe the infinite behaviour of the Hilbert functions, they do not guarantee that one can find a  $\mathcal{X}_2$  of approximately the same height and degree as  $\mathcal{X}_1$ , but the metric Bézout Theorem delivers useful results only if one can.

To evade this problem, one can use explicit estimates for the algebraic and arithmetic Hilbert functions that allow the argumentation for  $D$  and  $H$  not too big compared with the degree and height of  $\mathcal{X}_1$ , and therefore obtain  $\mathcal{X}_2$  and iteratively  $\mathcal{X}_3, \mathcal{X}_4, \dots$  with degree and height within a certain range. Of course, to obtain the desired results via this procedure, the estimates must relate the Hilbert functions to functions which are as close as possible to the leading term in the Hilbert-Samuel formulas.

For the algebraic Hilbert function of subvarieties of projective space, estimates of this kind have been obtained in [Ch], and [CP]. An upper bound for the arithmetic Hilbert function for subvarieties of projective space is given in [Ph]. This paper presents an alternative proof for the upper bound of the arithmetic Hilbert function and also a lower bound if the subvariety fulfills suitable conditions. The lower bound will only be given for irreducible varieties; the general case presents some additional problems, which however most probably are not very grave, and it maybe will be proved elsewhere.

The fundamental tool for the proofs presented here is the Theorem of Minkowski which allows to relate the arithmetic and algebraic Hilbert functions to lengths of shortest vector in the space of global sections of line bundles over arithmetic varieties, and thus in a certain sense reduces the arithmetic aspects of the problems to one dimensional sub spaces of arithmetic bundles.

The proofs of the bounds are related to and rely on a solution of the Problem of arithmetic interpolation which, roughly stated, asks, given two arithmetic subvarieties  $\mathcal{X}, \mathcal{Y}$  of projective space, under which conditions can one find a hypersurfaces of bounded height and degree that contains  $\mathcal{Y}$  and intersects  $\mathcal{X}$  properly? I am not yet sure whether the formula given for arithmetic interpolation, which is needed to prove the lower bound for the arithmetic Hilbert function given in this paper, solves

the interpolation problem in desirable generality; this will also be subject to further work.

Another important tool is the concept of locally complete intersection in projective space, which was already used in [CP] to obtain lower bounds for algebraic Hilbert functions and interpolation formulas in the algebraic context.

The concepts and results of the first part of this series on diophantine approximation will play no role in this paper though there are of course similar prerequisites and argumentations.

## 2 Algebraic Hilbert functions

In this section,  $k$  denotes an arbitrary field, and  $\mathbb{P}^t$  projective space over  $k$ .

**2.1 Definition** *A  $k$ - subscheme  $X$  of  $\mathbb{P}^t$  of pure codimension  $s$  is called a locally complete intersection of hypersurfaces  $H_1, \dots, H_s$  if there is an open subset  $U \subset \mathbb{P}^t$  such that*

$$X = \overline{H_1 \cap \dots \cap H_s \cap U}. \quad (4)$$

*Let  $Y \subset \mathbb{P}^t$  be an irreducible subvariety. A subvariety  $X$  of pure codimension  $s$  is called a locally complete intersection at  $Y$  if there are hypersurfaces  $H_1, \dots, H_s$  such that  $X$  consists of the irreducible components of  $H_1 \cap \dots \cap H_s$  that contain  $Y$ .*

### 2.2 Lemma

1. *If  $\mathcal{X} \subset \mathbb{P}^t$  is a locally complete intersection, and  $\mathcal{Y} \subset \mathcal{X}$  is a subvariety of codimension zero, then  $\mathcal{Y}$  is a locally complete intersection.*
2. *For any irreducible variety  $Y$ ; if  $X$  is a locally complete intersection at  $Y$ , then  $X$  is a locally complete intersection.*
3. *If  $X$  is a locally complete intersection at  $Y$ , and  $Z$  a subvariety that contains  $Y$  and intersects  $X$  properly, the union  $W$  of the components of  $X \cap Z$  that contain  $Y$  is a locally complete intersection at  $Y$ .*

**2.3 Proposition** *Let  $\mathcal{X} \subset \mathbb{P}^t$  be a subvariety of pure dimension  $s$  in  $\mathbb{P}^t$ , and denote by*

$$H_X(D) = \dim H^0(X, \mathcal{O}(D))$$

*the algebraic Hilbert function.*

1. For every  $D \in \mathbb{N}$ ,

$$H_X(D) \leq \deg X \binom{D+t-s}{t-s}.$$

2. If  $X$  is a locally complete intersection of hypersurfaces of degree  $D_1, \dots, D_s$ , then for  $D \geq \bar{D} := D_1 + \dots + D_s - s$ ,

$$H_X(D) \geq \deg X \binom{D - \bar{D} + t - s}{t-s}.$$

PROOF 1. [Ch], Theorem 1.

2. [CP], Corollary 3.

Let now  $Y \subset X \subset \mathbb{P}^t$  be algebraic subvarieties of dimension  $r$ , and  $s$  respectively, and  $V_n = V_n^X(Y)$  the  $n$ th infinitesimal neighbourhood of  $Y$  in  $X$ . Define

$$H_{V_n}(D) := \text{rk } \Gamma(V_n^X(Y), O(D)) = rk \Gamma(\mathbb{P}^t, O(D)) - rk I_{V_n}(D),$$

where  $I_{V_n}(D)$  are the global sections of  $O(D)$  that vanish on  $V_n^X(Y)$ .

## 2.4 Proposition

$$H_{V_n}(D) \leq \deg Y \binom{n+s-r-1}{s-r} \binom{D+r}{r}.$$

**2.5 Proposition** *Let  $X, Y$  be irreducible 0-dimensional subvarieties of  $\mathbb{P}^t$ , that are locally complete intersections of hypersurfaces of degree at most  $D$ . Then there is an  $f \in \Gamma(\mathbb{P}^t, O(tD))$  such that  $f|_X = 0$ , and  $f|_Y \neq 0$ .*

PROOF By proposition 2.3,

$$H_X(tD) \leq \deg X \binom{tD+0}{0} = \deg X,$$

and

$$H_{X \cup Y} \geq (\deg X + \deg Y) \binom{tD - d_1 - \dots - d_t + 0}{0} \geq \deg X + \deg Y.$$

Since  $rk \Gamma(\mathbb{P}^t, O(tD)) = \binom{tD+t}{t}$  this implies

$$\begin{aligned} rk I_{X \cup Y}(tD) &= \binom{tD+t}{t} - H_{X \cup Y} \leq \binom{tD+t}{t} - \deg X - \deg Y < \\ &\binom{tD+t}{t} - \deg X \leq \binom{tD+t}{t} - H_X(tD) = rk I_X(tD). \end{aligned}$$

Hence,  $I_X(tD)$  contains a vector not contained in  $I_{X \cup Y}(tD)$ , and thereby not contained in  $I_Y(tD)$ .

**2.6 Lemma** *Let  $k$  now be of characteristic zero,  $n \in \mathbb{N}$ , and  $v_1, \dots, v_n \in k^n$  such that for each  $i = 1, \dots, n$  the  $i$ th component of  $v_i$  is nonzero. Then there are  $m_1, \dots, m_n \in \mathbb{N}$  with  $m_i \leq n$  such that in*

$$v = m_1 v_1 + \dots + m_n v_n \in k^n$$

*no component is zero.*

**PROOF** The Lemma clearly holds for  $n = 1$ ; so assume it holds for  $n - 1$ . for  $v_1, \dots, v_n$  as in the Lemma, there are  $m_1, \dots, m_{n-1}$  with  $m_i \leq n - 1, i = 1, \dots, n - 1$  such that

$$w = m_1 v_1 + \dots + m_{n-1} v_{n-1}$$

has the first  $n - 1$  components not equal to zero. If also the last component of  $w$  is nonzero, choosing  $m_n = 0$  proves the Lemma. If the  $n$ th component of  $w$  equals zero, let  $u = v_n$ , and define  $w_j$  as the  $j$ th component of  $w$ , and  $u_j$  as the  $j$ th component of  $u$ . Further,  $k_j := w_j/u_j, j = 1, \dots, n - 1$ . As these are  $n - 1$  numbers there is an  $m_n \neq 0$  with  $m_n \leq n$  such that  $m_n \neq -k_j$  for all  $j = 1, \dots, n - 1$ . Consequently, for  $v_j$  the  $j$ th component of  $v = w + m_n u$ , we have  $v_j = w_j + m_n u_j$ . Thus, for  $j \leq n - 1$ , by the choice of  $m_n$  we have

$$v_j = w_j + m_n u_j \neq w_j - k_j u_j = 0,$$

and since  $w_n = 0$ , further  $v_n = m_n u_n \neq 0$ .

**2.7 Corollary** *Let  $X$  be an irreducible variety of dimension 0 in  $\mathbb{P}_{\mathbb{Q}}^t$ , and  $Y$  any 0-dimensional variety both of which are locally complete intersections of hypersurfaces of degree at most  $D$ . Then, there is an  $f \in \Gamma(\mathbb{P}^t, \mathcal{O}(tD))$  such that  $f|_X = 0$  but  $f$  is nonzero on every irreducible component of  $Y$ .*

**PROOF** Follows immediately from the previous two Lemmas.

**2.8 Proposition** *Let  $r < s \leq t$ , and  $X, Y$  irreducible subvarieties of  $\mathbb{P}^t$  of codimension  $r$ , and  $s$  respectively which are locally complete intersection of Hypersurfaces of degree at most  $D \geq t^{\frac{t}{s-r}}$ . Then there is an  $f \in \Gamma(\mathbb{P}^t, \mathcal{O}((t+1)D))$  such that  $f|_Y = 0$  and  $f|_X \neq 0$ .*

**PROOF** By Proposition 2.3,

$$H_X((t+1)D) \leq \deg X \binom{(t+1)D + t - s}{t - s},$$



and

$$H_{X \cap Y}((t+1)D) \geq (\deg X + \deg Y) \binom{D+t-r}{t-r}.$$

Using  $r < s$ , and  $D \geq t^{\frac{t}{s-r}}$  one easily calculates

$$\deg X \binom{(t+1)D+t-s}{t-s} < (\deg X + \deg Y) \binom{D+t-r}{t-r};$$

hence,

$$\mathrm{rk} I_{X \cap Y}((t+1)D) < \mathrm{rk} I_Y((t+1)D),$$

and consequently there is an  $f \in \Gamma(\mathbb{P}^t, \mathcal{O}((t+1)D))$  which is zero on  $Y$  but not zero on  $X \cup Y$ , hence not zero on  $X$ .

**2.9 Corollary** *Let  $s \leq t$ ,  $Y$  an irreducible subvariety of  $\mathbb{P}^t$  of codimension  $s$  respectively which is locally a complete intersection of Hypersurfaces of degree at most  $D \geq t^t$ , and  $X$  a subvariety each irreducible component of which has dimension smaller  $s$ , and is locally a complete intersection of degree at most  $D$ . Then there is an  $f \in \Gamma(\mathbb{P}^t, \mathcal{O}((t+1)D))$  such that  $f|_Y = 0$  and  $f|_X \neq 0$ .*

**PROOF** The proof is analogous to the one of the previous Corollary.

We make the notational convention that for positive numbers  $a, b$  occurring in a context of subvarieties of  $\mathbb{P}^t$ , the statement  $a \lesseqgtr b$  means that there are positive constants  $c_1, c_2$  depending only on  $t$ , and possibly on codimensions of subvarieties in the context such that  $a \leq c_1 b \leq c_2 a$ .

Using these interpolation formulas it is possible for each irreducible subvariety  $Y \subset \mathbb{P}^t$  to construct a chain consisting of locally complete intersections at  $Y$  of bounded degree:

**2.10 Proposition** *Let  $Y \subset \mathbb{P}^t$  be an irreducible variety of codimension  $s$ . There is a chain of subvarieties*

$$\mathbb{P}^t = X_0 \supset X_1 \supset \cdots \supset X_{s-1} \supset X_s \supset Y$$

*and hypersurfaces  $H_1, \dots, H_s$  of degree  $D_1 \leq \cdots \leq D_s$  such that each  $X_i, i = 0, \dots, s$  is the locally complete intersection of  $H_1, \dots, H_i$  at  $\mathcal{Y}$ , and the following conditions are fulfilled:*

1.

$$\deg X_i \lesseqgtr \deg X_{i-1} D_i, \quad H_Y(D_i - 1) \lesseqgtr \deg X_i \binom{D_i + t - i}{t - i},$$

*and if  $Y_i$  is an irreducible component of  $X_i$  with minimal degree, then*

$$\deg Y_i \lesseqgtr \deg X_i.$$

2.

$$\deg X_i \leq c(t, i)(\deg Y)^{\frac{i}{s}}.$$

3. With  $\bar{D}_i = D_1 + \dots + D_i - i$  for every  $i = 1, \dots, s-1$  the inequality  $D_{i+1} \geq 2(\bar{D}_i + 1)$  holds, and if  $D_{i+1} > 2(\bar{D}_i + 1)$  for some  $i \in \{1, \dots, s-1\}$ , then there exists no global section  $f$  of degree less than  $D_{i+1}$  that is zero on  $Y$  and nonzero on every irreducible component of  $X_i$ .

4. If  $Y$  is a locally complete intersection of hypersurfaces of degree at most  $D$ , each of the  $D_i$  may be chosen to be at most  $D$ .

PROOF For part 1, 2, and 3 see[CP] Theorem 2 or together with its proof. Part 3 is trivial.

### 3 Arithmetic varieties

I collect here several facts about arithmetic. Exept for number 3 below and Proposition 3.5 they all are either rather basic or can be found either in [SABK] or [BGS]. Let  $k$  be a number field with ring of integers  $\mathcal{O}_k$ , and  $\mathcal{X}$  a regular, flat, projective scheme of relative dimension  $d$  over  $\text{Spec} \mathbb{Z}$ . Under a subvariety of  $\mathcal{X}$  we will understand any integral subscheme that has at least one  $\bar{k}$ -valued point. For  $\mathcal{X} = \mathbb{P}_{\mathcal{O}_k}^t$  any nonempty intersection of zero sets of primitive elements in  $\Gamma(\mathbb{P}^t, \mathcal{O}(D))$ ,  $D > 0$  is a subvariety. A vector in  $\Gamma(\mathbb{P}^t, \mathcal{O}(D))$  is said to be primitive if it is not divisible by a nonunit in  $\mathcal{O}_k$ .

Let further  $\bar{\mathcal{L}}$  be an ample metrized line bundle on  $\mathcal{X}$ . Write  $X$  for the base extension of  $\mathcal{X}$  to  $\text{Spec } k$ , and for any embedding  $\sigma : k \hookrightarrow \mathbb{C}$  denote by  $X_\sigma$  the base extension of  $X$  to  $\mathbb{C}_\sigma$  as well as the  $\mathbb{C}$ -valued points of  $X$ . If clear from the context,  $X_\sigma$  will also be denoted by  $X$ . or  $\text{Spec } \mathbb{C}$  (also  $X_\infty$  for the last). For any effective cycle  $\mathcal{Y}$  on  $\mathcal{X}$ , i. e. a linear combination with positive coefficients of irreducible integral subschemes of  $\mathcal{X}$ , arithmetic intersection theory on  $\mathcal{X}$  (see e.g. [SABK]) enables define concept the height  $h(\mathcal{Y}) \in \mathbb{R}$  of  $\mathcal{Y}$ .

Furhter, the space of global section  $\Gamma(\mathcal{X}, \bar{\mathcal{L}})$  is a so called arithmetic bundle. Under an arithmetic bundle  $\bar{E}$  over  $\text{Spec } \mathcal{O}_k$  is to be understood a projective finitely generated  $\mathcal{O}_k$ -module  $E$  together with a hermitian product  $\langle \cdot | \cdot \rangle$  on  $E_\infty = E \otimes_{\mathcal{O}_k} \mathbb{C}$ . There are the following well known facts on heights and arithmetic degrees

1. Let  $\mathcal{X} = \mathbb{P}^t = \mathbb{P}(\mathbb{Z}^{t+1})$  be projective space of dimension  $t$ , over  $\text{Spec } \mathbb{Z}$  equipped with the line bundle  $\mathcal{L} = \mathcal{O}(1)$ . The canonical metric on  $\mathbb{C}^{t+1} = \mathbb{Z}^{t+1} \otimes_{\mathbb{Z}} \mathbb{C}$  induces a canonical metric on  $\mathcal{O}(1)$  (See e. g. [BGS], 4.1.1 for details). The height with respect to this metric is the so called Faltings height, and one has

$$h(\mathbb{P}^t) = \sigma_t, \quad \text{with} \quad \sigma_t = \frac{1}{2} \sum_{k=1}^p \sum_{m=1}^k \frac{1}{m}$$

the  $t$ th Stoll number.

Further, for any effective cycle  $\mathcal{Y}$  on  $\mathbb{P}^t$  the height of  $\mathcal{Y}$  is nonnegative. ([BGS], Proposition 3.2.4)

2. [BGS], Proposition 3.2.iv or [Ma1], Proposition 3.8.1.

For  $(\mathcal{X}, \bar{\mathcal{L}})$  arbitrary, let  $\mathcal{Y}$  be a subvariety of codimension  $p$  of  $\mathcal{X}$ , and  $f$  a global section of  $\mathcal{L}^{\otimes D}$  on  $\mathcal{X}$  whose restriction to  $Y$  is nonzero.

$$h(\mathcal{Y}.\text{div} f) = Dh(\mathcal{Y}) + \int_{X(\mathbb{C})} \log |f| \mu^{d-p} \delta_Y,$$

where  $\mu$  is the first chern form of  $\bar{\mathcal{L}}$ , and the integral is defined by resolution of singularities (See [SABK], II.1.2 for details).

3. Let  $\mathcal{X}$  be a subvariety of codimension  $p$  in projective space  $\mathbb{P}^t$ , and  $f \in \Gamma(\mathbb{P}^t, \mathcal{O}(D))$  a global section. Then,

$$\int_X \log |f| \mu^{t-p} - \deg X \int_{\mathbb{P}^t} \log |f| \mu^t \leq cD \deg X, \quad (5)$$

with  $c$  a positive constant only depending on  $t$ , and  $p$ .

**PROOF** Let  $X \# \text{div} f \supset \mathbb{P}^{2t+1}$  be the join of  $X$  and  $\text{div} f$  (see [Ma1], section 6 for the precise definition), and  $\mathbb{P}(\Delta) = \mathbb{P}^t \supset \mathbb{P}^{2t+1}$  the projective subspace corresponding to the diagonal in  $\mathbb{Z}^{2t+2}$ . If  $g_{\mathbb{P}(\Delta)}$  is a green current for  $\mathbb{P}(\Delta)$ , it follows from the proof of [BGS], Propostion 4.2.2 that

$$\int_X \log |f| \mu^{t-p} - \deg X \int_{\mathbb{P}^t} \log |f| \mu^t = -\frac{1}{2} \int_{X \# \text{div} f} g_{\mathbb{P}(\Delta)} + \frac{1}{2} \int_{\mathbb{P}^{2t+1}} g_{\mathbb{P}(\Delta)},$$

which by [BGS], Proposition 5.1.1 is at most

$$\deg(X \# \text{div} f)(\sigma_{p+1} + \sigma_{2t+1} - \sigma_{t-p} - \sigma_{t-1}) = D \deg X(\sigma_{p+1} + \sigma_{2t+1} - \sigma_{t-p} - \sigma_{t-1}).$$

**Remark:** In [Ma1] the number  $\int_X \log |f| \mu^{t-p} - \deg X \int_{\mathbb{P}^t} \log |f| \mu^t$  is called the algebraic distance  $D(X, \text{div} f)$  of  $X$  and the divisor corresponding to  $f$ . This concept however will not be needed in this paper.

4. Arithmetic Bézout Theorem ([BGS], Theorem 5.4.4): If  $p, q$  are natural numbers with  $p + q \leq t + 1$ , and  $\mathcal{X}, \mathcal{Y}$  effective cycles in  $\mathbb{P}^t$  of pure codimensions  $p$  and  $q$  intersecting properly, then

$$h(\mathcal{X}.\mathcal{Y}) \leq \deg Y h(\mathcal{X}) + \deg X h(\mathcal{Y}) + \left( \frac{t+1-p-q}{2} \right) \log 2 \deg X \deg Y.$$

5. For  $\bar{F}$  a one dimensional arithmetic bundle define the arithmetic degree

$$\widehat{\deg} \bar{F} := \sum_{s \in S} -\log |v|_s,$$

where  $v \in E$  is any nonzero element, and  $S$  denotes the set of all places of  $\mathcal{O}_k$ . For an arbitrary arithmetic bundle define

$$\widehat{\deg} \bar{E} := \widehat{\deg} \det \bar{F}.$$

If  $\mathcal{O} = \mathbb{Z}$ , then  $\widehat{\deg} \bar{E}$  is just minus the logarithm of the covolume of  $E$  in  $E \otimes_{\mathbb{Z}} \mathbb{R}$ .

If  $\bar{E}$  is an arithmetic bundle, and  $F \subset E$  a subbundle the metric on  $E_{\infty}$  induces a metric on  $F_{\infty}$ , and one obtains an arithmetic bundle  $\bar{F}$ . If one uses the canonical isomorphism of  $(E/F)_{\infty}$  with the orthogonal complement  $F_{\infty}^{\perp}$  of  $F_{\infty}$  in  $E_{\infty}$  the metric on  $E_{\infty}$  induces a metric on  $(E/F)_{\infty}$ , and one obtains an arithmetic bundle  $\overline{E/F}$ . Further, under suitable conditions on the metric on  $E$ , for example for the canonical metric on  $\mathbb{C}^n = \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{C}$ , the module  $G = E \cap F_{\infty}^{\perp}$  as rank  $\text{rk} E - \text{rk} F$ , and the metric on  $E_{\infty}$  induces a metric on  $G_{\infty} = F_{\infty}^{\perp}$ . We have

$$\widehat{\deg} \bar{G} \leq \widehat{\deg} \overline{E/F}. \quad (6)$$

**3.1 Theorem(Minkowski)** *Let  $\bar{M}$  be an arithmetic bundle  $\bar{M}$  over  $\text{Spec } \mathbb{Z}$ , and  $K \subset M_{\otimes \mathbb{Z}} \mathbb{R}$  any closed convex subset that is symmetric with respect to the origin, and fullfills*

$$\log \text{vol}(K) \geq -\widehat{\deg}(\bar{M}) + rkM \log 2.$$

*Then  $K \cap M$  contains a nonzero vector. In particular taking  $K$  as the cube with logarithmic length of edge  $-\frac{1}{rkM} \widehat{\deg}(\bar{M}) + \log 2$  centered at the origin, one sees that there is a non zero lattice point  $v \in M$  of logarithmic length*

$$\log |v| \leq -\frac{1}{rkM} \widehat{\deg}(\bar{M}) + \frac{1}{2} \log rkM.$$

Let now  $\mathbb{P}_{\mathbb{Z}}^t = \mathbb{P}(\mathbb{Z}^{t+1})$  be projective space of dimension  $t$ , and

$$E_D := \Gamma(\mathbb{P}^t, \mathcal{O}(D)).$$

As  $E_D = \text{Sym}^D E_1$ , which in turn equals the space of homogeneous polynomials of degree  $D$  in  $t+1$  variables, this lattice canonically carries the following metrics:

1. The subspace metric  $\text{Sym}^D E_1 \subset E_1^{\otimes D}$ .
2. The quotient metric  $E_1^{\otimes D} \rightarrow \text{Sym}^D E_1$  referred to as  $|f|_q$ .

3. The  $L^2$ -metric

$$|f|_{L^2(\mathbb{P}^t)}^2 = \int_{\mathbb{P}^t(\mathbb{C})} |f|^2 \mu^t,$$

where  $\mu$  is the Fubini-Study metric on  $\mathbb{P}^t$ .

4. The supremum metric

$$|f|_\infty = \sup_{x \in \mathbb{P}^t} |f_x|.$$

Among these metric the relations

$$\log |f|_\infty - \frac{D}{2} \sum_{m=1}^t \frac{1}{m} \leq \int_{\mathbb{P}_{\mathbb{C}}^t} \log |f| \mu^t \leq \log |f|_{L^2} \leq \log |f|_\infty. \quad (7)$$

hold ([BGS], (1.4.10) or [Ma1], Lemma 3.1.).

**3.2 Lemma** *Let  $I = (i_0, \dots, i_t)$  be a multiindex of order  $t + 1$ , and norm  $D$ , that is  $|I| = i_0 + \dots + i_t = D$ , and  $X^I$  the polynomial  $x_0^{i_0} \dots x_t^{i_t}$ . The set  $\{X^I \mid |I| = D\}$  forms a basis of  $\Gamma(\mathbb{P}_{\mathbb{Z}}^t, O(D))$ , and*

$$|X^I|_{L^2}^2 = |X^I|_q^2 \binom{D+t}{t} = \binom{D+t}{I}^{-1} = \frac{i_0! \dots i_t! t!}{(D+t)!} \leq 1.$$

Further there are constants  $c_1, c_2 > 0$  depending on  $t$  such that

$$-c_1 D \leq \log |X^I|_{L^2} = -\log \binom{D+t}{I} \leq -c_2 \log D,$$

$$-c_1 D \binom{D+t}{D} \leq \sum_I \log |X^I|_{L^2} \leq -c_2 \log D \binom{D+t}{D},$$

and

$$\sum_{\{I \mid |I|=D\}} \log |X^I|_{L^2} = -\sigma_t \frac{D^{t+1}}{(t+1)!} + O(D^t \log D).$$

**PROOF** The chain of equalities follows from [BGS], Lemma 4.3.6, and its proof. The estimates of  $\log \binom{D+t}{I}$  are easy calculations. From this the estimates on  $\sum_I \log |X^I|_{L^2}$  follow from the fact  $\dim \Gamma(\mathbb{P}^t, O(D)) = \binom{D+t}{t}$ . The last equality is the arithmetic Hilbert-Samuel formula for  $\mathbb{P}^t$ .

**3.3 Lemma** *Let  $f \in \Gamma(\mathbb{P}^t, O(D)), g \in \Gamma(\mathbb{P}^t, O(D'))$ . Then,*

$$\log |f|_{L^2} + \log |g|_{L^2} - c_2(\log D + \log D') \leq \log |fg|_{L^2} \leq$$

$$\log |f|_{L^2} + \log |g|_{L^2} + c_1(D + D') + \log \binom{D + D' + t}{t}.$$

PROOF Let  $I$  be any multiindex of order  $D$ . The vector  $X_I = x_0^{\otimes i_0} \otimes \cdots \otimes x_t^{\otimes i_t} \in E_1^{\otimes D}$  has length one, and by the previous Lemma

$$-c_2 \log D + \log |X_I| = -c_2 \log D \geq \log |X^I|_q + \log \binom{D+t}{t} \geq \log |X_I| - c_1 D.$$

Likewise, if  $I'$  is a multiindex of order  $D'$ ,

$$-c_2 \log D' \geq \log |X^{I'}|_q + \log \binom{D'+t}{t} \geq -c_1 D'.$$

Next,  $|X_I \otimes X_{I'}| = |X_I| |X_{I'}|$ . As the metric  $|\cdot|_q$  is the quotient metric of the one on  $E_1$ , we get

$$\begin{aligned} \log |X^I X^{I'}|_q &\leq \log |X_I X_{I'}| = 0 \leq \\ \log |X^I|_q + \log |X^{I'}|_q + c_1(D + D') &+ \log \binom{D+t}{t} + \log \binom{D'+t}{D'}, \end{aligned}$$

hence

$$\log |X^I X^{I'}|_{L^2} \leq \log |X^I|_{L^2} + \log |X^{I'}|_{L^2} + c_1(D + D') + \log \binom{D + D' + t}{t}.$$

Further, since  $|X^I|_{L^2} \leq 1$ ,

$$\log |X^I X^{I'}|_{L^2} \geq -c_2(\log D + \log D') \geq \log |X^I|_{L^2} + \log |X^{I'}|_{L^2} - c_2(\log D + \log D'),$$

proving the claim for  $f$ , and  $g$  monomials. The claim follows for general  $f$ , and  $g$  because the  $X^I, X_I$  form orthogonal bases.

Let  $X$  be a subvariety of pure dimension  $s$  in  $\mathbb{P}_{\mathbb{C}}^t$ . Then on

$$I_X(D) := \{f \in H^0(\mathbb{P}^t, \mathcal{O}(D)) \mid |f|_X = 0\},$$

there are the restrictions of the norms  $|\cdot|_{\text{sym}}$ ,  $|\cdot|_q$ , and  $|\cdot|_{L^2(\mathbb{P}^t)}$ , and on

$$F_X(D) = H^0(X, \mathcal{O}(D)), \tag{8}$$

there is the quotient norm  $|\cdot|_q$  induced by the quotient norm  $|\cdot|_q$  via the canonical quotient map

$$q_D : E_D \rightarrow F_X(D),$$

the  $L^2(\mathbb{P}^t)$ -norm

$$|\cdot|_{L^2(\mathbb{P}^t)} : F_D(X) \rightarrow \mathbb{R}, \quad \bar{f} \mapsto \inf_{q_D(f) = \bar{f}} \sqrt{\int_{\mathbb{P}^t} |f|^2 \mu^t} = \sqrt{\int_{\mathbb{P}^t} |f_X^\perp|^2 \mu^t} = |f_X^\perp|_{L^2(\mathbb{P}^t)},$$

with  $f_X^\perp$  the unique vector that is orthogonal to  $I_X(D)$ , and fullfills  $q_D(f_X^\perp) = \bar{f}$ , and the  $L^2(X)$ -norm

$$|\cdot|_{L^2(X)} : F_D(X) \rightarrow \mathbb{R}, \quad f \mapsto \int_X |f| \mu^p.$$

**3.4 Lemma** *Let  $X \subset \mathbb{P}_{\mathbb{C}}^t$  be a subvariety, and  $\bar{f} \in F_D(X), \bar{g} \in F_{D'}(X)$ . Then,*

$$\log |\overline{fg}|_{L^2(\mathbb{P}^t)} \leq \log |\bar{f}|_{L^2(\mathbb{P}^t)} + \log |\bar{g}|_{L^2(\mathbb{P}^t)} + c_1(D + D') + \log \binom{D + D' + t}{t} \leq$$

$$\log |\bar{f}|_{L^2(\mathbb{P}^t)} + \log |\bar{g}|_{L^2(\mathbb{P}^t)} + c_3(D + D'),$$

with  $c_1$  the constant from Lemma 3.2, and  $c_3 > c_1$  suitably chosen.

PROOF Let  $f \in \Gamma(\mathbb{P}^t, O(D)), g \in \Gamma(\mathbb{P}^t, O(D'))$  be representatives of  $\bar{f}, \bar{g}$ , and  $f = f_1 + f_2$  with  $f_1 \in I_X(D)$ , and  $f_2 \in I_X(D)^\perp$ , and likewise for  $g$ . Then,  $|\bar{f}|_{L^2(\mathbb{P}^t)} = |f_2|_{L^2}, |\bar{g}|_{L^2(\mathbb{P}^t)} = |g_2|_{L^2}$ . Let further  $\overline{fg}$  be represented by  $h = fg \in \Gamma(\mathbb{P}^t, O(D + D'))$  with decomposition  $h = h_1 + h_2$ . Then,  $h_2 = h - h_1 = (f_1g_1 + f_1g_2 + f_2g_1 - h_1) + f_2g_2$  with  $f_1g_1 + f_1g_2 + f_2g_1 - h_1 \in I_X(D + D')$ , and  $h_2 \in I_X(D + D')^\perp$ . Further,  $\overline{f_2g_2} = \overline{fg}$ . Consequently, by the previous Lemma,

$$\log |\overline{fg}|_{L^2(\mathbb{P}^t)} = \log |h_2|_{L^2} \leq \log |f_2g_2|_{L^2} \leq$$

$$\log |f_2|_{L^2} + \log |g_2|_{L^2} + c_1(D + D') + \log \binom{D + D' + t}{t} =$$

$$\log |\bar{f}|_{L^2(\mathbb{P}^t)} + \log |\bar{g}|_{L^2(\mathbb{P}^t)} + c_1(D + D') + \log \binom{D + D' + t}{t}.$$

There is a variant of the arithmetic Bézout Theorem given in 4 above, that gives a better estimate under certain conditions. Let  $\mathbb{P}^t$  be projective space over  $\mathcal{O}_k$ ,  $\mathcal{X} \subset \mathbb{P}^t$  an irreducible subvariety of codimension  $p$ , and  $f \in \Gamma(\mathbb{P}^t, O(D))_{\mathcal{O}_k}$  a global section that has nonzero restriction to  $\mathcal{X}$ .

**3.5 Proposition** *Under these assumptions*

$$h(\mathcal{X}.div f) \leq Dh(\mathcal{X}) + \deg X \log |f_X^\perp|_{L^2(\mathbb{P}^t)} + cD \deg X, \quad (9)$$

where  $c$  is a constant only depending on  $t$ , and the dimension of  $X$ .

PROOF Firstly, by number 2 above,

$$h(\mathcal{X}.div f) = Dh(\mathcal{X}) + \int_X \log |f| \mu^{t-p},$$

where  $\mu$  is the Fubini-Study metric on  $\mathbb{P}^t$ , or alternatively the first chern form of  $\mathcal{O}(1)$ . Next,  $f = f_x^\perp + g$  with  $g \in I_X(D)$ . Hence,

$$\int_X \log |f| \mu^{t-p} = \int_X \log |f_X^\perp| \mu^{t-p},$$

which by (5) is less or equal

$$\deg X \int_{\mathbb{P}^t} \log |f_X^\perp| \mu^t + cD \deg X \leq \log \int_{\mathbb{P}^t} \log |f_X^\perp| \mu^t + cD \deg X,$$

the last inequality following from (7).

**Remark:** The usual arithmetic Bézout Theorem gives no lower bound on  $h(\mathcal{X}, \mathcal{Y})$  for two properly intersecting cycles  $\mathcal{X}, \mathcal{Y}$ . However, in the situation of the Proposition it should be possible to give an estimate

$$h(\mathcal{X} \cdot \text{div} f) \geq Dh(\mathcal{X}) + \deg X \log |f_X^\perp| - c_1 D \deg X,$$

with  $c_1$  a constant only depending on  $t$ , and  $p$ .

## 4 Arithmetic Hilbert functions

For  $\mathcal{X} \subset \mathbb{P}^t$  a subvariety, let  $I_X(D)$ , and  $F_X(D)$  be as in (8), and  $G_X(D) = \Gamma(\mathbb{P}^t, O(D))_{\mathbb{Z}} \cap I_X(D)_{\infty}$  as in setction 3, number 5. If not stated otherwise, in this section  $|\cdot|$  will always denote the  $L^2$ -metric  $|\cdot|_{L^2(\mathbb{P}^t)}$ , on

$$E(D) = \Gamma(\mathbb{P}^t, O(D)), \quad I_X(D), \quad F_X(D), \quad G_X(D).$$

**4.1 Theorem** *Let  $\mathcal{X}$  be a subvariety of pure dimension  $s+1$  of  $\mathbb{P}_{\mathbb{Z}}^t$ , and denote by*

$$\hat{H}_{\mathcal{X}}(D) := \widehat{\deg}(\bar{F}_{\mathcal{X}}(D), |\cdot|_{L^2(\mathbb{P}^t)}), \quad \hat{\mathcal{H}}_{\mathcal{X}}(D) := \widehat{\deg}(\bar{F}_{\mathcal{X}}(D), |\cdot|_{L^2(X)})$$

*the arithmetic Hilbert functions.*

1. *For every  $D \in \mathbb{N}$ ,*

$$\hat{\mathcal{H}}_{\mathcal{X}}(D) \leq \deg X \left( Dh(\mathcal{X}) + \frac{1}{2}(\log \deg X + 2s \log D) \right) \binom{D+s}{s}.$$

2. *With a constant  $c_1$  only depending on  $t$ , and  $p$ ,*

$$\hat{H}_{\mathcal{X}}(D) \leq \left( Dh(\mathcal{X}) + c_1 \deg X D + \deg X \left( \frac{1}{2} \log \deg X + s \log D \right) \right) \binom{D+s}{D}.$$

*Hence for  $c_5 > c_1$ ,  $\deg X$  at most a fixed polynomial in  $D$ , and  $D$  sufficiently large,*

$$\hat{H}_{\mathcal{X}}(D) \leq (h(\mathcal{X}) + c_5 \deg X) D \binom{D+s}{D}.$$



3. With a positive constant  $c_4$  only depending on  $t$ , and  $p$ ,

$$\hat{H}_{\mathcal{X}}(D) \geq \widehat{\deg}(\bar{G}_{\mathcal{X}}(D)) \geq -\hat{H}_{\mathcal{X}}(D) + \hat{H}_{\mathbb{P}^t}(D) \geq \hat{H}_{\mathbb{P}^t}(D) -$$

$$\left( Dh(\mathcal{X}) + \deg XD(\sigma_{t-1} - \sigma_t) + \deg X \left( \frac{1}{2} \log \deg X + s \log D \right) \right) \binom{D+s}{D}.$$

For  $\deg X$  at most a fixed polynomial in  $D$ , and  $D$  sufficiently large, thus

$$\hat{H}_{\mathcal{X}}(D) \geq -(h(\mathcal{X}) + c_5 \deg X) D \binom{D+s}{s}.$$

4. There are constants  $c_6, c_7 > 0, N \in \mathbb{N}$  only depending on  $t$ , and  $p$  such that if  $\mathcal{X}$  is an irreducible locally complete intersection of hypersurfaces of degree  $D_1, \dots, D_{t-s}$ , then for  $D \geq N\bar{D} := N(D_1 + \dots + D_{t-s} - s)$ , the inequality

$$\hat{H}_{\mathcal{X}}(D) \geq (c_6 h(\mathcal{X}) - c_7 \deg X) D \binom{D+s}{s}$$

holds.

PROOF 1. Let  $\mathcal{X} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_n$  be the decomposition into irreducible components. We use complete induction on  $n$ . If  $n = 1$ , i. e.  $\mathcal{X}$  is irreducible, let  $f \in F_{\mathcal{X}}(D)$  be nonzero. Then, by section 3, number 2 and (7),

$$\begin{aligned} 0 \leq h(\mathcal{X} \cdot \text{div}(f)) &= Dh(\mathcal{X}) + \int_X \log |f| \mu^s \leq Dh(\mathcal{X}) + \log \int_X |f| \mu^s \\ &= Dh(\mathcal{X}) + \log |f|_{L^2(X)}. \end{aligned}$$

Hence,

$$\log |f|_{L^2(X)} \geq -Dh(\mathcal{X})$$

for every nonzero vector  $f \in F_D(\mathcal{X})$ . By the Theorem of Minkowski,

$$-\widehat{\deg}(\bar{F}_D, |\cdot|_{L^2(X)}) \geq -rk F_D Dh(\mathcal{X}) - \frac{rk F_D}{2} \log rk F_D.$$

By Proposition 2.3.1,  $-rk F_D \geq -\deg X \binom{D+s}{s}$ ; hence the above is greater or equal

$$-\deg X \left( Dh(\mathcal{X}) + \frac{1}{2} (\log \deg X + 2s \log D) \right) \binom{D+s}{s},$$

which proves the claim for  $n = 1$ . Assume now the claim has been proved for  $n - 1$ . We have the surjective restriction map

$$\varphi : F_{\mathcal{X}}(D) \rightarrow F_{\mathcal{X}_1 \cup \dots \cup \mathcal{X}_{n-1}}(D),$$

and  $\bar{F}_{\mathcal{X}_1 \cup \dots \cup \mathcal{X}_{n-1}}(D) = \overline{F_{\mathcal{X}}/\ker \varphi}$ . Hence,

$$\hat{\mathcal{H}}_{\mathcal{X}}(D) = \widehat{\deg} \bar{F}_{\mathcal{X}}(D) = \widehat{\deg} \overline{\ker \varphi} + \widehat{\deg} \bar{F}_{\mathcal{X}_1 \cup \dots \cup \mathcal{X}_{n-1}}(D),$$

which by induction hypothesis is at most

$$\widehat{\deg} \overline{\ker \varphi} + \sum_{i=1}^{n-1} \deg X_i \left( D \sum_{i=1}^{n-1} h(\mathcal{X}_i) + \frac{1}{2} \left( \log \left( \sum_{i=1}^{n-1} \deg X_i \right) + 2s \log D \right) \right) \binom{D+s}{s}.$$

Next,  $\ker \varphi$  maps injectively to  $F_{\mathcal{X}_n}(D)$ , hence as above for every  $f \in \ker \varphi$

$$\log |f| \geq \log |f_{X_n}^\perp| \geq -Dh(\mathcal{X}_n),$$

and

$$\text{rk} \ker \varphi \leq H_{X_n}(D) \leq \deg X_n \binom{D+s}{s},$$

which together implies

$$-\deg X_n \left( Dh(\mathcal{X}_n) + \frac{1}{2} (\log \deg X_n + 2s \log D) \right) \binom{D+s}{s},$$

finishing the proof.

2. Let  $\mathcal{X} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_n$  be the decomposition into irreducible components. If  $n = 1$ , i. e.  $\mathcal{X}$  is irreducible, denote by  $f_X^\perp$  the orthogonal projection of  $f \in \Gamma(\mathbb{P}^t, \mathcal{O}(D))$  modulo  $I_{X_i}(D)$ . Clearly  $|f_X^\perp| \leq |f|$ . Further by Proposition 3.5,

$$\deg X \log |f_X^\perp|_{L_2(\mathbb{P}^t)} \geq h(\mathcal{X} \cdot \text{div}(f)) - Dh(\mathcal{X}) - cD \deg X \geq -Dh(\mathcal{X}) - cD \deg X.$$

Consequently,

$$\log |f|_{L^2(\mathbb{P}^t)} \geq -D \frac{h(\mathcal{X})}{\deg X} - cD.$$

As  $f \in I_D(\mathcal{X})^\perp \setminus \{0\}$  was arbitrary, the claim follows for  $n = 1$  in the same way as part one.

Assume now the claim has been proved for  $n - 1$ . With the notations of the proof of part one,

$$\hat{H}_{\mathcal{X}}(D) = \widehat{\deg} \overline{\ker \varphi} + \hat{H}_{\mathcal{X}_1 \cup \dots \cup \mathcal{X}_{n-1}}(D),$$

which by induction hypothesis is less or equal than,

$$\widehat{\deg} \overline{\ker \varphi} + \left( D \sum_{i=1}^{n-1} h(\mathcal{X}_i) + c_1 \sum_{i=1}^{n-1} \deg X_i D + \sum_{i=1}^{n-1} \deg X_i \left( \frac{1}{2} \log \sum_{i=1}^{n-1} \deg X_i + s \log D \right) \right) \binom{D+s}{D}.$$

Similarly as in the proof of part 1, one proves

$$\widehat{\deg} \overline{\ker \varphi} \leq \left( Dh(\mathcal{X}_n) + c_1 \deg XD + \deg X_n \left( \frac{1}{2} \log \deg X_n + s \log D \right) \right) \binom{D+s}{D},$$

finishing the proof of part 2.

Part three follows from a fact about volumes of orthogonal complements in lattices. For arithmetic interpolation the following generalization of the second inequality in part three will be needed.

**4.2 Lemma** *Let  $\mathcal{Y} \subset \mathcal{X}$  be subvarieties of  $\mathbb{P}^t$  of codimensions  $s$ , and  $r$ , and  $I_{\mathcal{X}/\mathcal{Y}}^\perp(D) = I_{\mathcal{Y}}(D)_{\mathbb{Z}} \cap I_{\mathcal{X}}(D)^\perp$  the orthogonal complement of  $I_{\mathcal{X}}(D)$  in  $I_{\mathcal{Y}}(D)$  as in section 3, number 5. Then,*

$$\begin{aligned} -\widehat{\deg} \bar{I}_{\mathcal{X}/\mathcal{Y}}^\perp(D) &\leq -\widehat{\deg} \bar{I}_{\mathcal{Y}}(D) - \widehat{\deg} \bar{I}_{\mathcal{X}}(D) + 2\hat{H}_{\mathbb{P}^t}(D) \leq \\ &\left( Dh(\mathcal{X}) + c_1 \deg XD + \deg X \left( \frac{1}{2} \log \deg X + s \log D \right) \right) \binom{D+t-s}{D} + \\ &\left( Dh(\mathcal{Y}) + c_1 \deg YD + \deg Y \left( \frac{1}{2} \log \deg Y + s \log D \right) \right) \binom{D+t-s}{D}, \end{aligned}$$

where  $c_1, c_5$  are the constants from Theorem 4.1.2, and  $c_2$  is the constant  $c_1$  from Lemma 3.2.

PROOF Renormalizing the norm of the basis  $\{X^I \mid |I| = D\}$  of  $\Gamma(\mathbb{P}^t, O(D))$  by the factors  $\binom{D+t}{I}$ , one obtains a norm  $|\cdot|_{aux}$  on  $\Gamma(\mathbb{P}^t, O(D))_{\mathbb{R}}$  that takes integer values on  $\Gamma(\mathbb{P}^t, O(D))_{\mathbb{Z}}$ , hence on  $I_{\mathcal{Y}}(D)$ . We have

$$-\widehat{\deg} \overline{\Gamma(\mathbb{P}^t, O(D))} = -\hat{H}_{\mathbb{P}^t}(D), \quad -\widehat{\deg} \overline{\Gamma(\mathbb{P}^t, O(D))}_{aux} = 0,$$

and since by Lemma 3.2  $\log |X^I|^{-1} = \binom{D+t}{I} \geq 0$  for every  $I$ , for any arithmetic subbundle  $\bar{M} \subset \overline{\Gamma(\mathbb{P}^t, O(D))}$ ,

$$-\widehat{\deg} \bar{M} \leq -\widehat{\deg} \bar{M}_{aux} \leq -\widehat{\deg} \bar{M} + \hat{H}_{\mathbb{P}^t}(D). \quad (10)$$

By [Be], Proposition 1.ii,

$$-\widehat{\deg} \bar{I}_{\mathcal{X}/\mathcal{Y}}^\perp(D)_{aux} \leq -\widehat{\deg} \bar{I}_{\mathcal{Y}}(D)_{aux} - \widehat{\deg} \bar{I}_{\mathcal{X}}(D)_{aux}.$$

Hence, by (10),

$$-\widehat{\deg} I_{\mathcal{X}/\mathcal{Y}}^\perp(D)_{aux} \leq -\widehat{\deg} \bar{I}_{\mathcal{Y}}(D)_{aux} - \widehat{\deg} \bar{I}_{\mathcal{X}}(D)_{aux} \leq$$

$$-\widehat{\deg} \bar{I}_{\mathcal{Y}}(D) - \widehat{\deg} \bar{I}_{\mathcal{X}}(D) + 2\hat{H}_{\mathbb{P}^t}(D),$$

giving the first inequality. The second inequality then simply follows from

$$\hat{H}_{\mathbb{P}^t}(D) = \widehat{\deg} \bar{I}_{\mathcal{X}}(D) + \hat{H}_{\mathcal{X}}(D),$$

which holds by definition, the corresponding equality for  $\mathcal{Y}$ , and Theorem 4.1.2.

**PROOF OF THEOREM 4.1.3:** The first inequality is (6), and the second inequality immediately follows from the previous Lemma taking the higher dimensional variety equal to  $\mathbb{P}^t$ .

**4.3 Lemma** *Let  $\bar{M} \subset \bar{N} \subset \bar{D}(D) = \overline{\Gamma(\mathbb{P}^t, O(D))}$  be arithmetic subbundle. Then,*

$$\widehat{\deg} \overline{E(D)/M} \geq \widehat{\deg} \overline{N/M};$$

*in particular*

$$\hat{H}_{\mathcal{Y}}(D) \geq 0$$

*for every subvariety  $\mathcal{Y}$  of  $\mathbb{P}^t$ .*

**PROOF** Let  $q : E(D) \rightarrow E(D)/M$  be the canonical projection. Since  $B = \{X^I \mid |I| = D\}$  forms a basis of  $E(D)$ , there is a subset  $\bar{B} \subset B$  such that  $q(\bar{B})$  forms a basis of  $N/M$ , and because of  $|q(X^I)| \leq |X^I| \leq 1$ , the Lemma follows.

## 4.1 Arithmetic Interpolation

For the rest of the paper constants  $c_1, c_2, \dots$  that appear without saying will always be positive and only depending on  $t$  and possibly the dimension of some sub variety of  $\mathbb{P}^t$  appearing in the respective context.

**4.4 Proposition** *Let  $\mathcal{X}, \mathcal{Y}$  be subvarieties of  $\mathbb{P}^t$  of pure codimensions  $r$ , and  $s$  with  $r < s$ , and assume that  $\mathcal{Y}$  is irreducible, and  $\mathcal{X}$  is a locally complete intersection of hypersurfaces of degrees  $D_1 \leq \dots \leq D_r$ . Set  $\bar{D} := D_1 + \dots + D_r - r$ . Let further  $\mathcal{X} = \mathcal{X}_1 \cup \dots \cup \mathcal{X}_n$  be the decomposition into irreducible components,  $d = \min_{i=1, \dots, n} \deg X_i$ , and  $c_3 = \sqrt[s-r]{\frac{2^{t+1-r}(t-r)!}{(t-s)!}}$ . Then, for  $D \geq \max(2\bar{D}, c_3 \sqrt[s-r]{\frac{\deg Y}{d}})$ , and every  $i = 1, \dots, n$ ,*

$$\begin{aligned} rk I_Y(D) / (I_{X_i}(D) \cap I_Y(D)) &\geq d \binom{D - \bar{D} + t - r}{t - r} - \deg Y \binom{D + t - s}{t - s} \\ &\geq \frac{d}{2} \binom{D - \bar{D} + t - r}{t - r} \geq \frac{d}{2} \binom{D/2 + t - r}{t - r}. \end{aligned}$$

and there is an  $f \in I_Y(D)$  that is nonzero on  $\mathcal{X}_i$ , for every  $i = 1, \dots, n$ , and fullfills

$$\log |f| \leq 2 \frac{(Dh(\mathcal{X}) + c_5 \deg XD) \binom{D+t-r}{D}}{\frac{d}{2} \binom{D/2+t-r}{t-r}} + \frac{(Dh(\mathcal{Y}) + c_1 \deg YD + \deg Y(\frac{1}{2} \log \deg Y + s \log D)) \binom{D+t-s}{D}}{\frac{d}{2} \binom{D/2+t-r}{t-r}},$$

where  $c_1, c_5$  are the constants from Theorem 4.1.2.

PROOF The first set of inequalities follow from the two parts of Proposition 2.3, and the equality

$$\operatorname{rk} I_Y(D) / (I_{X_i}(D) \cap I_Y(D)) = H_{X \cup Y}(D) - H_Y(D).$$

With  $I_{\mathcal{X}_i/\mathcal{Y}}(D) := I_Y(D)_Z \cap (I_X(D) \cap I_Y(D))^\perp$ , by the Theorem of Minkowski, for every  $i = 1, \dots, n$ , there is an  $f_i \in I_Y(D)$  that is nonzero on  $\mathcal{X}_i$  such that

$$\log |f_i| \leq -\frac{\widehat{\deg} \bar{I}_{\mathcal{X}_i/\mathcal{Y}}^\perp(D)}{\operatorname{rk} I_Y(D) / (I_{X_i}(D) \cap I_Y(D))} + \frac{1}{2} \log \operatorname{rk} I_Y(D) / (I_{X_i}(D) \cap I_Y(D)). \quad (11)$$

By Lemma 2.6, there are nonzero numbers  $l_1, \dots, l_n \in \mathbb{N}$  with  $l_i \leq n, i = 1, \dots, n$  such that

$$f := \sum_{i=1}^n l_i f_i$$

is nonzero on each irreducible component of  $\mathcal{X}$ . Thus, the inequality (11) together with Lemma 4.2 applied to the varieties  $\mathcal{X}_i, \mathcal{X}_i \cap \mathcal{Y}$ , implies

$$\begin{aligned} \log |f| &\leq \max_{i=1, \dots, n} \log |f_i| + 2 \log n \\ &\leq 2 \frac{(Dh(\mathcal{X}) + c_1 \deg XD + \deg X(\frac{1}{2} \log \deg X + r \log D)) \binom{D+t-r}{D}}{\frac{d}{2} \binom{D/2+t-r}{t-r}} \\ &\quad + \frac{(Dh(\mathcal{Y}) + c_1 \deg YD + \deg Y(\frac{1}{2} \log \deg Y + s \log D)) \binom{D+t-s}{D}}{\frac{d}{e} \binom{D/2+t-r}{t-r}} \\ &\quad + 2 \log \deg X \\ &\leq 2 \frac{(Dh(\mathcal{X}) + c_1 \deg XD) \binom{D+t-r}{D}}{\frac{d}{2} \binom{D/2+t-r}{t-r}} \\ &\quad + \frac{(Dh(\mathcal{Y}) + c_5 \deg YD + \deg Y(\frac{1}{2} \log \deg Y + s \log D)) \binom{D+t-s}{D}}{\frac{d}{e} \binom{D/2+t-r}{t-r}}, \end{aligned}$$

which was to be proved.

**Remark:** If  $\mathcal{Y} \subset \mathcal{X}$ , the stronger inequality

$$\log |f| \leq \frac{(Dh(\mathcal{X}) + c_5 \deg XD) \binom{D+t-r}{D}}{\frac{d}{2} \binom{D/2+t-r}{t-r}} + \frac{(Dh(\mathcal{Y}) + c_1 \deg YD + \deg Y(\frac{1}{2} \log \deg Y + s \log D)) \binom{D+t-s}{D}}{\frac{d}{2} \binom{D/2+t-r}{t-r}},$$

holds for  $f$ .

Let now  $\mathcal{Y} \subset \mathbb{P}_{\mathcal{O}_k}^t$  be an irreducible subvariety,  $H_1, \dots, H_s$  hypersurfaces of degrees  $D_1 \leq \dots \leq D_s$  that contain  $Y$ , and

$$\mathbb{P}^t = X_0 \supset X_1 \supset \dots \supset X_s \supset Y = \mathcal{Y}_k \quad (12)$$

a chain of subvarieties such that each  $X_i$  is the locally complete intersection of  $H_1, \dots, H_i$  at  $Y$ . These data are supposed to fulfill the properties stated in Proposition 2.10. It is evident from the proof of this Proposition that the hypersurfaces and subvarieties may be chosen to be defined over  $\text{Spec} \mathcal{O}_k$ . Also, the concept of locally complete intersection and locally complete intersection at  $\mathcal{Y}$  transfer to the arithmetic case, and Lemma 2.2 still holds. Thus we have  $H_i = (\mathcal{H}_i)_k$ ,  $X_i = (\mathcal{X}_i)_k$  with  $\mathcal{H}_i, \mathcal{X}_i$  hypersurfaces and locally complete intersections at  $\mathcal{Y}$  defined over  $\text{Spec} \mathcal{O}_k$ . However, the  $\mathcal{X}_i$  are not uniquely determined by  $Y$ , their choice rather being quite restrictive, if the gaps between the different numbers  $D_i$  (each  $D_i$  is uniquely determined by  $X_{i-1}$ ) are big. For example for  $X_1 = H_1 = V(f_1)$ , the vector can only be chosen among the vectors in  $I_Y(D_1)$  which has small dimension, if  $D_1$  is small compared with  $\sqrt{s \deg Y}$ ; if  $s = 2$  its dimension is one if  $D_1$  is small. These considerations lead to the following Definition: For  $m \in \mathbb{N}$ , the chain (12) is called  $m$ -stable if

$$m^{s-1} D_1 \geq m^{s-2} D_2 \geq \dots \geq m D_{s-1} \geq D_s.$$

Every chain can be dissected into parts

$$\mathcal{X}_0 \supset \dots \supset \mathcal{X}_{i_1} \supset \mathcal{X}_{i_1+1} \supset \dots \supset \mathcal{X}_{i_2} \supset \dots \supset \mathcal{X}_{i_{k-1}} \supset \mathcal{X}_{i_{k-1}+1} \supset \dots \supset \mathcal{X}_{i_k} = \mathcal{X}_s \supset Y,$$

such that the subchains

$$\mathcal{X}_0 \supset \dots \supset \mathcal{X}_{i_1}, \quad \mathcal{X}_{i_1+1} \supset \dots \supset \mathcal{X}_{i_2}, \quad \mathcal{X}_{i_{k-1}} \supset \mathcal{X}_{i_{k-1}+1} \supset \dots \supset \mathcal{X}_{i_k} = \mathcal{X}_s \supset \mathcal{Y},$$

are  $m$ -stable, and

$$D_{i_1} < m D_{i_1+1}, \dots, D_{i_{k-1}} < m D_{i_{k-1}+1}.$$

The  $m$ -stable sub chains above are called the  $m$ -stable parts of the chain

**4.5 Proposition** *Let  $m, n$  be natural numbers  $n \geq 4$ , and  $\mathcal{Y} \subset \mathbb{P}^t$  an irreducible subvariety of codimension  $s$ . Further*

$$\mathbb{P}^t = \mathcal{X}_0 \supset \mathcal{X}_1 \cdots \supset \mathcal{X}_{i_1} \supset \cdots \supset \mathcal{X}_{i_{k-1}} \supset \cdots \supset \mathcal{X}_{i_k-1} \supset \mathcal{X}_{i_k} = \mathcal{X}_s \supset \mathcal{Y}$$

*subvarieties with the properties in Proposition 2.10, with  $m$ -stable parts*

$$\mathcal{X}_0 \supset \cdots \supset \mathcal{X}_{i_1}, \quad \cdots, \quad \mathcal{X}_{i_{k-1}+1} \supset \cdots \supset \mathcal{X}_{i_k}.$$

*Then, there is a chain of subvarieties*

$$\mathbb{P}^t = \bar{\mathcal{X}}_0 \supset \cdots \supset \bar{\mathcal{X}}_s \supset \mathcal{Y},$$

*a chain of irreducible subvarieties*

$$\mathbb{P}^t = \bar{\mathcal{Y}}_0 \supset \bar{\mathcal{Y}}_1 \supset \cdots \supset \bar{\mathcal{Y}}_{s-1} \supset \bar{\mathcal{Y}}_s = \mathcal{Y},$$

*numbers  $\bar{D}_1 \leq \cdots \leq \bar{D}_s$ , and hypersurfaces  $\bar{\mathcal{H}}_1, \dots, \bar{\mathcal{H}}_s, \bar{\mathcal{G}}_1, \dots, \bar{\mathcal{G}}_s$ , such that each  $\bar{\mathcal{X}}_i, i = 0, \dots, s$  is the locally complete intersection of  $\bar{\mathcal{H}}_1, \dots, \bar{\mathcal{H}}_i$  at  $\mathcal{Y}$ , each  $\bar{\mathcal{Y}}_i$  intersects  $\bar{\mathcal{G}}_{i+1}$  properly, and  $\bar{\mathcal{Y}}_{i+1}$  is an irreducible component of  $\bar{\mathcal{Y}}_i \cdot \bar{\mathcal{G}}_{i+1}$ , and with certain positive constants  $c_1, c_2, c_7, c_8$  only depending on  $t, s$ , and  $i$ , but independent of  $m$ , and  $n$ , the following conditions are fulfilled:*

1. *For  $j = 1, \dots, k$ :  $\bar{D}_{i_j} = D_{i_j}$ ,  $\bar{\mathcal{X}}_{i_j} = \mathcal{X}_{i_j}$ , and  $\bar{\mathcal{Y}}_{i_j}$  is an irreducible component of  $\mathcal{X}_{i_j}$ , and there is no global section of degree less than  $D_{i_j+1}$  that is zero on  $\mathcal{Y}$ , but nonzero on  $\mathcal{Y}_{i_j}$ .*

*For  $i \neq i_j$  for all  $j = 1, \dots, k$ :  $nc_1 D_i \leq \bar{D}_i \leq 4^{i-i_j} c_2 n D_i$ . Hence, the chains*

$$\bar{\mathcal{X}}_0 \supset \cdots \supset \bar{\mathcal{X}}_{i_1}, \quad \cdots, \quad \bar{\mathcal{X}}_{i_{k-1}+1} \supset \cdots \supset \bar{\mathcal{X}}_{i_k}$$

*modulo constants are  $mn$ -stable.*

2.

$$\deg \bar{X}_i \leq n^i \deg X_i, \quad \deg \bar{X}_i \geq \deg \bar{X}_{i-1} \bar{D}_i,$$

$$n^{t-i} H_Y(\bar{D}_i - 1) \leq \deg \bar{X}_i \binom{\bar{D}_i + t - i}{t - i},$$

*and if  $d_i$  denotes the minimum of the degrees of the irreducible components of  $\bar{X}_i$ ,*

$$d_i \leq \deg \bar{X}_i, \quad \text{and} \quad \deg \bar{Y}_i \leq \deg \bar{X}_i.$$

3.

$$\deg \bar{X}_i \leq c(t, i) n^i (\deg Y)^{\frac{i}{s}}.$$

4. For every  $j = 1, \dots, k-1$ , the restriction maps in the chain

$$F_{Y_{i_j}}(D_{i_{j+1}} - 1) \rightarrow F_{Y_{i_{j+1}}}(D_{i_{j+1}} - 1) \rightarrow \dots \rightarrow F_Y(D_{i_{j+1}} - 1),$$

all are bijections.

5. For  $j = 0, \dots, k-1$ , and  $i_j < i < i_{j+1}$

$$\begin{aligned} h(\bar{\mathcal{X}}_i) &\leq c_8(s, i)(h(\mathcal{X}_{i_{j+1}} + c_5 \deg X_{i_{j+1}})\bar{D}_i^{i-i_j} + \\ &\quad c_7(s, i)(h(\mathcal{X}_{i_{j+1}}) + c_5 \deg X_{i_{j+1}})\bar{D}_i^{i-i_{j+1}}, \end{aligned}$$

and

$$\begin{aligned} h(\bar{\mathcal{Y}}_i) &\leq c_8(s, i)(h(\mathcal{Y}_{i_{j+1}} + c_5 \deg Y_{i_{j+1}})\bar{D}_i^{i-i_j} + \\ &\quad c_7(s, i)(h(\mathcal{Y}_{i_{j+1}}) + c_5 \deg Y_{i_{j+1}})\bar{D}_i^{i-i_{j+1}}, \end{aligned}$$

where  $c_5$  is the constant from Theorem 4.1.2.

PROOF The existence of  $\bar{\mathcal{X}}_i$  and  $\bar{\mathcal{Y}}_i$  is trivial for  $i = 0$ ; so assume it is proved for  $i < s$ .

If  $i = i_j - 1$  for some  $j = 1, \dots, k$ , take  $\bar{\mathcal{X}}_{i+1} = \mathcal{X}_{i+1}$ ,  $\bar{D}_{i+1} = \max(D_{i+1}, 2\bar{D}_i)$  and  $\bar{\mathcal{Y}}_{i+1}$  as an irreducible component of  $\mathcal{X}_{i+1}$  such that the restriction map

$$F_{\bar{\mathcal{Y}}_{i+1}}(D_{i_{j+1}} - 1) \rightarrow F_Y(D_{i_{j+1}} - 1)$$

is injective; such a component exists by Proposition 2.10.3. The claims of the Proposition then are trivial or follow from Proposition 2.10.

Let now  $i_j \leq i < i_{j+1}$  for some  $j$ . The idea of the proof is of course to apply the interpolation formula in Proposition 4.4. Because this interpolation formula only works if the degrees of the involved varieties, and the degree of the global section to construe fullfill the inequalities in the assumption, we firstly have to compare degrees of varieties and the numbers  $D_{i_{j+1}}, \dots, D_{i-1}, \bar{D}_{i_{j+1}}, \dots, \bar{D}_{i-1}$ .

Firstly by Proposition 2.10,

$$\deg X_{i_{j+1}} \leq cD_{i_{j+1}} \deg X_{i_{j+1}-1} \leq \dots \leq \deg X_i c^{i_{j+1}-i} \prod_{l=i+1}^{i_{j+1}} D_l,$$

thus by assumption

$$\deg X_{i_{j+1}} \leq \deg X_i c^{i_{j+1}-i} m^{(i_{j+1}-i)(i_{j+1}-i+1)/2} D_{i+1}^{i_{j+1}-i}. \quad (13)$$



Further, by induction hypothesis,

$$\deg \bar{X}_i \geq c\bar{D}_i \deg \bar{X}_{i-1} \geq \cdots \geq c^{i-i_j} \deg \bar{X}_{i_j} \prod_{l=i_j+1}^i \bar{D}_l \quad (14)$$

$$\geq c^{i-i_j} \deg X_{i_{j+1}} \frac{1}{D_{i_{j+1}}} \prod_{l=i_j}^i \bar{D}_l \geq \cdots \geq c^{i-i_j} \deg X_i \frac{\prod_{l=i_j}^i \bar{D}_l}{\prod_{l=i_{j-1}}^i D_l} \quad (15)$$

$$\geq (cc_2n)^{i-i_j} \deg X_i. \quad (16)$$

If  $d_i$  is the degree of the irreducible component of  $\bar{X}_i$  with minimal degree, by induction hypothesis  $\deg \bar{X}_i \leq \bar{c}d_i$ , which together with (13), and (16) implies

$$\begin{aligned} D_{i+1} &\geq {}^{i_{j+1}-i}\sqrt{\frac{\deg X_{i_{j+1}}}{d_i}} {}^{i_{j+1}-i}\sqrt{\frac{(cc_2n)^{i-i_j}}{\bar{c}c^{i_{j+1}}m^{(i_{j+1}-i)(i_{j+1}-i-1)/2}}} = \\ &c_1 {}^{i_{j+1}-i}\sqrt{n^{i-i_j}} {}^{i_{j+1}-i}\sqrt{\frac{\deg X_{i_{j+1}}}{d_i}}. \end{aligned}$$

Thus, with  $c_3$  the constant from Proposition 4.4, and

$$\bar{D}_{i+1} := \max(2(\bar{D}_i + \cdots + \bar{D}_1), n([c_3c_1] + 1)D_{i+1}),$$

we have

$$\bar{D}_{i+1} \geq c_3n {}^{s-i}\sqrt{\frac{\deg X_{i_{j+1}}}{d_i}}. \quad (17)$$

Also,  $\bar{D}_{i+1} \geq n([c_3c_1] + 1)D_{i+1}$ , and induction hypothesis for part 1 of the Proposition, and Proposition 2.10.3,

$$2(\bar{D}_i + \cdots + \bar{D}_1) \leq 4^{i-i_j} 2c_2n(D_i + \cdots + D_1) \leq c_24^{i+1-i_j}nD_{i+1},$$

hence  $\bar{D}_{i+1} \leq nD_{i+1}\max([c_3c_1 + 1], 4^{i+1-i_j}c_2)$ , and  $\bar{D}_{i+1}$  fullfills the inequalities of part one of the Propsotion.

Now, by (17), and Proposition 4.4, there is a nonzero vector  $f \in I_{\bar{Y}_{i_{j+1}}}(\bar{D}_{i+1})$  that has nonzero restriction to every irreducible component of  $\bar{\mathcal{X}}_i$ , and fullfills

$$\begin{aligned} \log |f| &\leq \frac{(\bar{D}_{i+1}h(\bar{\mathcal{X}}_i) + c_5\bar{D}_{i+1} \deg \bar{X}_i) \binom{\bar{D}_{i+1}+t-i}{t-i}}{\frac{d_i}{2} \binom{\bar{D}_{i+1}/2+t-i}{t-i}} \\ &+ \frac{(h(\bar{\mathcal{Y}}_{i_{j+1}}) + c_4 \deg \bar{Y}_{i_{j+1}}) \bar{D}_{i+1} \binom{\bar{D}_{i+1}+t-i_{j+1}}{t-i_{j+1}}}{\frac{d_i}{2} \binom{\bar{D}_{i+1}/2+t-i}{t-i}} \\ &+ \frac{\deg \bar{Y}_{i_{j+1}} (\frac{1}{2} \log \deg \bar{Y}_{i_{j+1}} + i_{j+1} \log \bar{D}_{i+1}) \binom{\bar{D}_{i+1}+t-i_{j+1}}{t-i_{j+1}}}{\frac{d_i}{2} \binom{\bar{D}_{i+1}/2+t-i}{t-i}} \end{aligned}$$

Since  $\deg \bar{Y}_{i_{j+1}}$  is at most a fixed polynomial in  $\bar{D}_{i+1}$ ,

$$\log |f| \leq \frac{(h(\bar{\mathcal{X}}_i) + c_5 \deg \bar{X}_i) \bar{D}_{i+1} \binom{\bar{D}_{i+1} + t - i}{t - i}}{\frac{d_i}{2} \binom{\bar{D}_{i+1}/2 + t - i}{t - i}} + \quad (18)$$

$$\frac{(h(\bar{\mathcal{Y}}_{i_{j+1}}) + c_5 \deg \bar{Y}_{i_{j+1}}) \bar{D}_{i+1} \binom{\bar{D}_{i+1} + t - i_{j+1}}{t - i_{j+1}}}{\frac{d_i}{2} \binom{\bar{D}_{i+1}/2 + t - i}{t - i}}. \quad (19)$$

Take  $\bar{\mathcal{H}}_{i+1} := \operatorname{div} f$ , and  $\bar{\mathcal{X}}_{i+1}$  as the union of the irreducible components of  $\bar{\mathcal{X}}_i \cdot \bar{\mathcal{H}}_{i+1}$  that contain  $\bar{\mathcal{Y}}_{i_{j+1}}$ . By Lemma 2.2.3,  $\bar{\mathcal{X}}_{i+1}$  is a locally complete intersection at  $\bar{\mathcal{Y}}$ . To estimate  $h(\bar{\mathcal{X}}_{i+1})$ , just use the sharp arithmetic Bézout Theorem (3.5), and apply the induction hypothesis. More concretely, (3.5) together with (18) firstly implies

$$\begin{aligned} h(\bar{\mathcal{X}}_{i+1}) &\leq h(\bar{\mathcal{X}}_i \cdot \bar{\mathcal{H}}_{i+1}) \leq \bar{D}_{i+1} h(\bar{\mathcal{X}}_i) + \deg \bar{X}_i \log |f|_{L^2(\mathbb{P}^t)} + c_9 \deg \bar{X}_i \bar{D}_{i+1} \leq \\ &\quad \bar{D}_{i+1} (h(\bar{\mathcal{X}}_i) + c_9 \deg \bar{X}_i) + \\ &\quad \frac{\deg \bar{X}_i}{d_i 2^{t-i}} \left( c_6 \bar{D}_{i+1} (h(\bar{\mathcal{X}}_i) + c_5 \deg \bar{X}_i) + (\bar{D}_{i+1} h(\bar{\mathcal{Y}}_{i_{j+1}}) + c_5 \deg \bar{Y}_{i_{j+1}}) c_3 \bar{D}_{i+1}^{i-i_{j+1}} \right). \end{aligned}$$

Next, by induction hypothesis for part 2,  $d_i \geq \deg \bar{X}_i / \bar{c}_1$ , and the above is less or equal

$$\begin{aligned} &\bar{D}_{i+1} (h(\bar{\mathcal{X}}_i) + c_9 \deg \bar{X}_i) + \\ &\bar{c}_1 \left( c_6 \bar{D}_{i+1} (h(\bar{\mathcal{X}}_i) + c_5 \deg \bar{X}_i) + (\bar{D}_{i+1} h(\bar{\mathcal{Y}}_{i_{j+1}}) + c_5 \deg \bar{Y}_{i_{j+1}}) c_3 \bar{D}_{i+1}^{i-i_{j+1}} \right) \leq \\ &((1 + \bar{c}_1 c_6) h(\bar{\mathcal{X}}_i) + (\bar{c}_1 c_5 + c_9) \deg \bar{X}_i) \bar{D}_{i+1} + c_3 (h(\bar{\mathcal{Y}}_{i_{j+1}}) + c_5 \deg \bar{Y}_{i_{j+1}}) \bar{D}_{i+1}^{i-i_{j+1}+1}. \end{aligned}$$

Using the fact  $\deg \bar{X}_i \leq \bar{D}_{i+1} \cdots \bar{D}_{i_j+1} \deg X_{i_j+1} \leq \bar{D}_i^{i-i_j-1} \deg X_{i_j}$ , and again by induction hypothesis, this time for part five of the proposition, one gets

$$\begin{aligned} h(\bar{\mathcal{X}}_{i+1}) &\leq c_8(s, i) (1 + \bar{c}_1 c_6) (h(\bar{\mathcal{X}}_{i_j}) + c_5 \deg X_{i_j}) \bar{D}_{i+1} \bar{D}_i^{i-i_j-1} + \\ &\quad c_7(t, i) (1 + \bar{c}_1 c_6) (h(\bar{\mathcal{Y}}_{i_{j+1}}) + c_5 \deg \bar{Y}_{i_{j+1}}) \bar{D}_{i+1} \bar{D}_i^{i-i_{j+1}} + \\ &\quad c_3 (h(\bar{\mathcal{Y}}_{i_{j+1}}) + c_5 \deg \bar{Y}_{i_{j+1}}) \bar{D}_{i+1}^{i-i_{j+1}+1} + (\bar{c}_1 c_5 + c_9) \bar{D}_{i+1}^{i-i_j-1} \deg X_{i_j} \\ &\leq c_8(t, i+1) (h(\bar{\mathcal{X}}_{i_j}) + c_5 \deg Y_{i_j}) \bar{D}^{i-i_j} + \\ &\quad c_7(t, i+1) (h(\bar{\mathcal{Y}}_{i_{j+1}}) + c_5 \deg \bar{Y}_{i_{j+1}}) \bar{D}_{i+1}^{i-i_{j+1}+1}, \end{aligned}$$

with

$$c_8(t, i+1) = c_8(t, i) (1 + \bar{c}_1 c_6) + (\bar{c} + \frac{c_9}{c_5}), \quad c_7(t, i+1) = c_7(t, i) (1 + \bar{c}_1 c_6) + c_3,$$

proving part 5 for  $\bar{\mathcal{X}}_i$ . Part 1 for  $\bar{\mathcal{X}}_i$  has already been seen above. For part 3,

$$\begin{aligned} \deg \bar{X}_{i+1} &\leq \bar{D}_{i+1} \deg \bar{X}_i \leq \bar{c}_1 D_{i+1} n \deg \bar{X}_i \leq \bar{c}_1 \bar{c}_2 n^{i+1} D_{i+1} \deg X_i \leq \\ &\bar{c}_1 \bar{c}_2 \bar{c}_3 n^{i+1} \deg X_{i+1}, \end{aligned}$$

the first inequality being Bézout's Theorem, the second following from part 1, the third from the induction hypothesis, and the fourth from Proposition 2.10.1. Part 3 for  $\bar{X}_{i+1}$  thus follows from Proposition 2.10.2.

To construct and prove parts 3, and 5 for  $\bar{\mathcal{Y}}_{i+1}$ , apply induction hypothesis of part 2, to see that  $\deg \bar{Y}_i \leq \bar{c}_1 \deg \bar{X}_i$ , and use the inequalities  $\deg \bar{X}_i \leq \bar{c}_2 d_i$  from above, and  $\deg Y_{i+1} \leq \deg X_{i+1}$ , to derive

$$\bar{D}_{i+1} \geq \bar{c}_3 \sqrt[s-i]{\frac{\deg Y_{i+1}}{\deg Y_i}}.$$

After multiplying  $\bar{D}_{i+1}$  by a fixed constant, if necessary, one can apply the same procedure as for  $\bar{\mathcal{X}}_{i+1}$  above.

Part 2 follows in the same way as part 3 above

Finally, part four follows from the choice of  $\bar{\mathcal{Y}}_{i_j}$  subject to the condition in Proposition 2.10.3.

## 4.2 The lower bound

**4.6 Lemma** *Let  $\mathcal{Y} \subset \mathcal{X} \subset \mathbb{P}^t$  be irreducible subvarieties of codimensions  $s$ , and  $s-1$  respectively.*

1. *For every  $f \in I_{\mathcal{Y}}(D)$ , let again  $f_X^\perp$  be the orthogonal projection of  $f$  modulo  $I_X(D)$ . Then, with some positive constant  $c$ ,*

$$\log |f_X^\perp|_{L^2(X)} = \log |f|_{L^2(X)} \geq h(\mathcal{Y}) - Dh(\mathcal{X}).$$

$$\log |f|_{L^2(\mathbb{P}^t)} \geq \log |f_X^\perp|_{L^2(\mathbb{P}^t)} \geq \frac{h(\mathcal{Y})}{\deg X} - D \frac{h(\mathcal{X})}{\deg X} - cD$$

*for every  $f \in I_{\mathcal{Y}}(D)$  with  $f_X^\perp \neq 0$ .*

2. *If further  $\mathcal{X}$  is a locally complete intersection of hypersurfaces of degrees  $D_1 \leq \dots \leq D_{s-1}$ , and  $\mathcal{Y}$  is an irreducible component of a proper intersection of  $\mathcal{X}$  with a hypersurface of degree  $D_s$ , then, for  $\bar{D} = D_1 + \dots + D_{s-1} - s + 1$ , and  $D \geq \max(2\bar{D}, 3(t-s+1)D_s)$ ,*

$$-\widehat{\deg}(\bar{I}_{\mathcal{Y}}(D)/\bar{I}_{\mathcal{X}}(D)) \geq \frac{1}{3(t+t-s)!} h(\mathcal{Y}) D^{t-s+1} - (h(\mathcal{X}) + c \deg X) D^{t-s+2}.$$

PROOF 1. Let  $\mathcal{Z} = \mathcal{X}.\text{div}f$ . Then, clearly  $\mathcal{Y}$  is a subvariety of codimension 0 of  $\mathcal{Z}$ , and hence  $h(\mathcal{Y}) \leq h(\mathcal{Z})$ . By section 3, number 2, and (7),

$$h(\mathcal{Y}) \leq h(\mathcal{Z}) \leq Dh(\mathcal{X}) + \int_X \log |f| \mu^{t+1-s} \leq Dh(\mathcal{X}) + \log |f|_{L^2(X)},$$

and the first claim follows. Further, by 3, number 3

$$\begin{aligned} \int_X \log |f| \mu^{t+1-s} &= \int_X \log |f_X^\perp| \mu^{t+1-s} \leq \deg X \int_{\mathbb{P}^t} \log |f_X^\perp| \mu^t + cD \deg X \leq \\ &c \deg X D + \deg X \log |f_X^\perp|_{L^2(\mathbb{P}^t)}, \end{aligned}$$

proving the second claim.

2. By Propostion 2.3,

$$\begin{aligned} \text{rk}I_Y(D)/I_X(D) &= H_X(D) - H_Y(D) \geq \\ \deg X \binom{D - \bar{D} + t - s + 1}{t - s + 1} &- \deg Y \binom{D + t - s}{t - s}. \end{aligned}$$

Since  $\deg Y \leq D_s \deg X$ , by the choice of  $D$  this is greater or equal

$$\frac{1}{3(t-s+1)!} \deg X D^{t+1-s}.$$

Further,

$$\text{rk}I_Y(D)/I_X(D) = H_X(D) - H_Y(D) \leq H_X(D) \leq \deg X \binom{D + t + 1 - s}{t + 1 - s},$$

by Proposition 2.3.

By part one, and the Theorem of Minkowski, thus

$$\begin{aligned} -\widehat{\deg}(\bar{I}_Y(D)/\bar{I}_X(D)) &\geq \left( \frac{h(\mathcal{Y})}{\deg X} - D \frac{h(\mathcal{X})}{\deg X} - cD \right) (\text{rk}I_Y(D)/I_X(D)) \\ &\quad - \frac{1}{2} (\text{rk}I_Y(D)/I_X(D)) \log(\text{rk}I_Y(D)/I_X(D)), \end{aligned}$$

which because of the two estimates of  $\text{rk}I_Y(D)/I_X(D)$  is greater or equal

$$\begin{aligned} \frac{1}{3(t+1-s)!} h(\mathcal{Y}) D^{t-s+1} &- (h(\mathcal{X}) + c \deg X) D^{t-s+2} \\ &- \frac{1}{2} (\text{rk}I_Y(D)/I_X(D)) \log(\text{rk}I_Y(D)/I_X(D)). \end{aligned}$$

The claim thus follows from the upper estimate of  $\text{rk}I_Y(D)/I_X(D)$  above, if one enlarges  $c$ , and keeps in mind, that  $\deg X$  is at most  $D^{s-1}$ , hence  $\log \deg X \leq (s-1) \log D$ .

**4.7 Lemma** *Let  $\mathcal{Y}$  be an irreducible subvariety of codimension  $s$  of  $\mathbb{P}^t$  that is locally a complete intersection of hypersurfaces of degrees  $D_1 \leq \dots \leq D_s$ , define  $\bar{D} = D_1 + \dots + D_{s-1} - s + 1$ , and assume that Theorem 4.1.4 holds for subvarieties of codimension at most  $s - 1$ . Then, there are constants  $e, c_1, c_6 > 0, e \leq 6(t + 1 - s)$  only depending on  $t$  such that for  $D = [eD_s]$  the arithmetic bundle  $\bar{F}_{\mathcal{Y}}(D)$  contains a non zero lattice point  $f$  with*

$$\log |f| \leq -c_1 \frac{h(\mathcal{Y})}{\deg Y} D + c_6 D.$$

PROOF For  $m, n$  be natural numbers  $\geq 4$ , and

$$\begin{aligned} \mathcal{X}_0 \supset \mathcal{X}_1 \supset \dots \supset \mathcal{X}_s \supset \mathcal{Y}, \quad \bar{\mathcal{X}}_0 \supset \bar{\mathcal{X}}_1 \supset \dots \supset \bar{\mathcal{X}}_s \supset \mathcal{Y}, \\ \bar{\mathcal{Y}}_0 \supset \bar{\mathcal{Y}}_1 \supset \dots \supset \bar{\mathcal{Y}}_{i_{k-1}} \supset \bar{\mathcal{Y}}_{i_{k-1}+1} \supset \dots \supset \bar{\mathcal{Y}}_{s-1} \supset \mathcal{Y} \end{aligned} \quad (20)$$

and  $D_1, \dots, D_s, \bar{D}_1, \dots, \bar{D}_s$  be as in Propositions 2.10, and 4.5, where the first chain is assumed to be  $m$ -stable, and the second chain  $mn$ -stable consequently. Further  $nD_l \leq \bar{D}_l \leq nc_9 D_l$  for  $l = 1 \dots, s$ . Write simply  $i$  for  $i_{k-1}$ ; this means that the chain  $\mathcal{X}_{i+1} \supset \dots \supset \mathcal{X}_{s-1} \supset \mathcal{Y}$  is  $m$ -stable, but the chain  $\mathcal{X}_i \supset \dots \supset \mathcal{X}_{s-1} \supset \mathcal{Y}$  is not. Set  $\bar{D} := D_1 + \dots + D_{s-1} - s + 1$ ,  $M = 3(t + 1 - s)([D_s/\bar{D}] + 1)$ , and  $D = M\bar{D}$ . Then,  $D \geq \max(2\bar{D}, 3(t + 1 - s)D_s)$ , hence by Lemma 4.6,

$$-\widehat{\deg} \bar{I}_{\mathcal{Y}}(D)/\bar{I}_{\bar{\mathcal{Y}}_{s-1}}(D) \geq (c_1 h(\mathcal{Y}) - Dh(\bar{\mathcal{Y}}_{s-1}))D^{t+1-s} - c_2 D \deg X D^{t-s+1}. \quad (21)$$

The variety  $\bar{\mathcal{Y}}_{s-1}$  by Proposition 4.5 fullfills

$$h(\bar{\mathcal{Y}}_{s-1}) \leq c_3(h(\bar{\mathcal{Y}}_i) + c_5 \deg \bar{Y}_i) \bar{D}_{s-1}^{s-1-i} + c_4(h(\mathcal{Y}) + c_5 \deg Y) \bar{D}_{s-1}^{-1}, \quad (22)$$

and each of the constants  $c_1, c_2, c_3, c_4, c_5$  is independent of  $m$  and  $n$ . Choose now  $m = 4t$ ,  $n = \left\lceil \frac{6tmc_4}{c_1} \right\rceil + 1$ , and  $C := \frac{2c_3}{c_4} \left( \frac{c_9 n}{3(t+1-s)} \right)^{s-i}$ .

CASE ONE:  $h(\mathcal{Y}) \geq CD^{s-i}h(\bar{\mathcal{Y}}_i)$ . In this case,

$$D \geq 3(t + 1 - s)D_s \geq \frac{3(t + 1 - s)}{nc_9} \bar{D}_s \geq \frac{3(t + 1 - s)}{nc_9} \bar{D}_{s-1} \geq \frac{3t}{nc_9} \frac{n}{m} D_s \geq \frac{1}{mc_9} D, \quad (23)$$

Further, by (21),

$$\begin{aligned} \hat{H}_{\mathcal{Y}}(D) &= \sigma_t \frac{D^{t+1}}{(t+1)!} - \widehat{\deg} \bar{I}_{\mathcal{Y}}(D) \\ &= \sigma_t \frac{D^{t+1}}{(t+1)!} - \widehat{\deg}(\bar{I}_{\mathcal{Y}}(D)/\bar{I}_{\bar{\mathcal{Y}}_{s-1}}(D)) - \widehat{\deg} \bar{I}_{\bar{\mathcal{Y}}_{s-1}}(D) \\ &\geq (c_1 h(\mathcal{Y}) - Dh(\bar{\mathcal{Y}}_{s-1}))D^{t+1-s} - c_2 \deg \bar{Y}_{s-1} D^{t-s+2} + \hat{H}_{\bar{\mathcal{Y}}_{s-1}}(D), \end{aligned}$$

which by Lemma 4.3 is greater or equal

$$(c_1 h(\mathcal{Y}) - Dh(\bar{\mathcal{Y}}_{s-1}))D^{t+1-s} - c_2 \deg \bar{Y}_{s-1} D^{t-s+2}.$$

By (22),

$$2c_4 h(\mathcal{Y}) - \bar{D}_{s-1} h(\bar{\mathcal{Y}}_{s-1}) \geq c_4 h(\mathcal{Y}) - c_5 \deg Y - c_3 (h(\bar{\mathcal{Y}}_i) + c_5 \deg \bar{Y}_i) \bar{D}_{s-1}^{s-i},$$

which by the choice of  $C$ , the assumption  $-Ch(\bar{\mathcal{Y}}_i)D^{s-i} \geq -h(\mathcal{Y})$ , and the inequality  $\deg Y \leq \frac{c}{n^t} \deg \bar{Y}_i \bar{D}_{i+1} \cdots \bar{D}_s \leq \frac{c}{n^t} \deg \bar{Y}_i \bar{D}_{s-1}^{s-i}$  together with (23) is greater or equal

$$c_4 h(\mathcal{Y}) - (c_5 + cc_5/n^t) \deg Y - c_4 h(\mathcal{Y})/2 = c_4 h(\mathcal{Y})/2 - (c_5 + cc_5/n^t) \deg Y.$$

By the choice of  $n$ , hence

$$\begin{aligned} \hat{H}_{\mathcal{Y}}(D) &\geq (c_1 h(\mathcal{Y}) - Dh(\bar{\mathcal{Y}}_{s-1}))D^{t+1-s} - c_2 \deg \bar{Y}_{s-1} D^{t+2-s} \\ &\geq \frac{c_1}{2c_4} (2c_4 h(\mathcal{Y}) - \bar{D}_{s-1} h(\bar{\mathcal{Y}}_{s-1}))D^{t+1-s} - c_2 \deg \bar{Y}_{s-1} D^{t+2-s} \\ &\geq \frac{c_1}{2} (h(\mathcal{Y}) - \frac{c_5 n^t + cc_5}{n^t c_4} \deg Y) D^{t+1-s} - c_2 \deg \bar{Y}_{s-1} D^{t-s+2}. \end{aligned}$$

Because  $\deg \bar{Y}_{s-1}$ , by Proposition 2.10 is at most a constant times  $\frac{\deg Y}{D^s}$ , consequently,

$$\hat{H}_{\mathcal{Y}}(D) \geq (\frac{c_1}{2} h(\mathcal{Y}) - c_6 D \deg Y) D^{t-s+1}.$$

On the other hand, by Proposition 2.3,

$$\text{rk} F_D(Y) \leq \deg Y \binom{D+t-s}{t-s} \leq \deg Y D^{t-s}.$$

The Theorem of Minkowski thus implies the existence of a vector  $f \in F_D(\mathcal{Y})$  with

$$\log |f| \leq -\frac{c_1 h(\mathcal{Y})}{2 \deg Y} D + c_6 D,$$

proving the Lemma in case one with  $e = M$ .

CASE TWO:  $h(\mathcal{Y}) \leq CD^{s-i} h(\bar{\mathcal{Y}}_i)$ . If the first chain in (20) is  $m$ -stable, that is  $\bar{\mathcal{Y}}_i = \mathbb{P}^t$ , and thereby  $h(\mathcal{Y}) \leq CD^s h(\mathbb{P}^t) = CD^s \sigma_t$ , then there is a multiindex  $I$  with  $|I| = D$  such that  $X^I \in \Gamma(\mathbb{P}^t, O(D))$  is nonzero on  $\mathcal{Y}$ , and

$$\log |X^I| \leq -c \log D \leq -\frac{c \log D}{\sigma_t} \frac{h(\mathcal{Y})}{CD^s} \leq -C_1 \frac{h(\mathcal{Y})}{\deg Y} m^{s(s+1)/2} D,$$

the last inequality following from the fact  $m$ -stability of the first chain in (20) together with Proposition 4.5.2 implies  $\deg Y \geq C_2 m^{s(s+1)/2} D^s$ . Hence, the Lemma follows in this case also with  $e = M$ .

If on the other hand  $i > 0$ , that is the chain is not  $m$ -stable, we have

$$D_i < \frac{1}{m} D_{i+1} \geq \frac{1}{m^2} D_{i+2} \geq \cdots \geq \frac{1}{m^{s-i}} D_s.$$

Consequently, if  $\tilde{D} := [D_{i+1}/2]$ ,

$$\tilde{D} \geq 2D_{i+1} \geq \frac{2}{m} D_{i+2} \geq \cdots \geq \frac{2}{m^{s-i-2}} D_{s-1} \geq \frac{2}{m^{s-i-2}i} \bar{D} = \frac{2}{m^{s-i-2}iM} D.$$

By hypothesis, Theorem 4.1.4 holds for  $\bar{\mathcal{Y}}_i$ , that is

$$\widehat{\deg F_D}(\bar{\mathcal{Y}}_i) = \hat{H}_{\bar{\mathcal{Y}}_i}(\tilde{D}) \geq (c_6 h(\bar{\mathcal{Y}}_i) - c_7 \deg Y_i) \tilde{D}^{t+1-i}, \quad (24)$$

and since  $\bar{\mathcal{Y}}_i$  locally is an intersection of hypersurfaces of degrees  $D_1, \dots, D_i$ , and  $D_1 + \dots + D_i - i \leq iD_i < \frac{i}{m} D_{i+1} \leq \frac{2i}{m} \tilde{D}$ , Proposition 2.3.2 implies

$$\mathrm{rk} F_{\bar{\mathcal{Y}}_i}(\tilde{D}) = H_{\bar{\mathcal{Y}}_i}(\tilde{D}) \geq \deg \bar{\mathcal{Y}}_i \binom{\tilde{D}(1 - \frac{2i}{m}) + t - i}{t - i} \geq c_{10} \left(1 - \frac{2i}{m}\right)^{t-i} \deg \bar{\mathcal{Y}}_i \tilde{D}^{t-i}.$$

By the Theorem of Minkowski  $F_{\mathcal{Y}_i}(\tilde{D})$  thus contains a non zero vector  $f$  with

$$\begin{aligned} \log |g| &\leq \frac{(-c_6 h(\mathcal{Y}_i) + c_7 \deg \bar{\mathcal{Y}}_i) \tilde{D}^{t+1-i}}{c_{10} (1 - \frac{2i}{m})^{t-i} \deg \bar{\mathcal{Y}}_i \tilde{D}^{t-i}} \\ &\quad + \frac{1}{2} \log(c_{10} (1 - \frac{2i}{m})^{t-i} \deg Y_i \tilde{D}^{t-i}). \end{aligned}$$

Since  $\deg \bar{\mathcal{Y}}_i$  is less or equal a fixed polynomial in  $\tilde{D}$ , and  $\frac{2i}{m} \leq \frac{1}{2}$ , the assumption  $h(\mathcal{Y}) \leq CD^{s-i} h(\bar{\mathcal{Y}}_i)$  together with the inequality  $D \leq m^{s-i-2} i M / 2 \tilde{D}$  implies

$$\log |g| \leq -c_{11} \frac{h(\mathcal{Y})}{\deg Y} \tilde{D} + c_{12} \tilde{D},$$

for some positive constants  $c_{11}, c_{12}$ . By Proposition 4.5.4, in the chain

$$H^0(\bar{\mathcal{Y}}_i, \mathcal{O}(\tilde{D})) \rightarrow \cdots \rightarrow H^0(\bar{\mathcal{Y}}_{s-1}, \mathcal{O}(\tilde{D})) \rightarrow H^0(\bar{\mathcal{Y}}_s, \mathcal{O}(\tilde{D})) \rightarrow H^0(\mathcal{Y}, \mathcal{O}(\tilde{D}))$$

each map is injective, and thus the image  $f$  of  $g$  in  $F_{\mathcal{Y}}(\tilde{D})$  is nonzero, and clearly  $\log |f| \leq \log |g|$ . Thus, the claim follows with  $e = \frac{2}{i m^{s-i-2}}$ .

**4.8 Lemma** *Let  $Y$  be a subvariety of  $\mathbb{P}^t$ , which is a locally complete intersection of hypersurfaces of degree  $D_1 \leq \dots \leq D_s$ ,  $\bar{D} := D_1 + \dots + D_s - s$ , and  $f \in \Gamma(\mathbb{P}^t, \mathcal{O}(\deg f))$  a global section whose restriction to  $Y$  is not a zero divisor, and let  $V_k(Y.\text{div} f)$  be the  $k$ th infinitesimal neighbourhood of  $Y.\text{div} f$  in  $Y$ . If  $I_{V_k \deg f}(D)$  denotes the elements of  $F_Y(D)$  that vanish on  $V_k \deg f(Y.\text{div} f)$ , and  $E_k(D) \subset I_{V_k \deg f}(D)$  the multiples of  $f^k$  in  $F_Y(D)$ , then for  $D \geq 2\bar{D} + 2k \deg f$ ,*

$$\text{rk } I_{V_k \deg f}(D) \geq \text{rk } E_k(D) \geq \deg Y \binom{[D/2] + t - s}{t - s}.$$

PROOF Follows immediately from the the fact that  $f$  and thereby  $f^k$  is not a zero divisor in  $F_Y$ , and the formula

$$H_Y(D - k \deg f) \geq \deg Y \binom{D - k \deg f - \bar{D} + t - s}{t - s},$$

from Proposition 2.3.

PROOF OF THEOREM 4.1.4: The proof is by complete induction on the codimension of  $\mathcal{X}$ . If the codimension is 0, then  $\mathcal{X} = \mathbb{P}^t$ , and by Lemma 3.2,

$$\hat{H}_{\mathcal{X}}(D) \geq \frac{\sigma_t}{2} \frac{D^{t+1}}{(t+1)},$$

if  $D \geq N$  with  $N$  a number only depending on  $m$ . Assume now the Theorem is proved for subvarieties of codimension at most  $s-1$ .

By Lemma 4.7, there are constants  $e, c_1, c_2 > 0$   $e \leq 6(t+1-s)$ , a such that with  $\tilde{D} = eD_s$ , there is a non zero  $f \in F_{\tilde{D}}(\mathcal{X})$  such that

$$\log |f| \leq -c_1 \frac{h(\mathcal{X})}{\deg X} \tilde{D} + c_2 \tilde{D}.$$

Assume first that  $\tilde{D} \geq 3\bar{D}$ , let  $D \geq \tilde{D}$ , and  $m, n \in \mathbb{N}, m < \tilde{D}$  be such that  $D = n\tilde{D} + m$ . Further define  $\mathcal{Y}$  as the divisor in  $\mathcal{X}$  corresponding to  $f$ . With  $E_{[n/2]}(D) \subset I_{V_{[n/2]\tilde{D}}}(D)$  as in Lemma 4.8, this Lemma implies

$$\text{rk } E_{[n/2]}(D) \geq \deg Y \binom{[D/2] + t - s}{t - s}.$$

Further any  $g \in E_{[n/2]}(D)$  may be written as  $g = f^{[n/2]}h$  with  $h \in F_{n\tilde{D}/2+m}$ . Choosing a basis of  $E_{[n/2]}(D)$  in such a way that each basis element equals  $f^{[n/2]}h$  with some  $h$  such that  $\log |h| \leq 0$ , Lemma 3.3 implies that these basis elements have logarithmic length at most

$$[n/2] \log |f| + cD \leq [n/2] \left( -c_1 \frac{h(\mathcal{X})}{\deg X} \tilde{D} + c_2 \tilde{D} \right) + cD \leq$$



$$-c_1 \frac{h(\mathcal{X})}{\deg X} \frac{D}{2} + c_2 \frac{D}{2} + cD \leq -c_1 \frac{h(\mathcal{X})}{\deg Y} \frac{D}{2} + c_3 D.$$

Hence

$$\begin{aligned} -\widehat{\deg} \bar{E}_{[n/2]} &\leq \deg X \binom{[D/2] + t - s}{t - s} \left( -c_1 \frac{h(\mathcal{Y})}{\deg X} \frac{D}{2} + c_3 D \right) = \\ &(-c_6 h(\mathcal{X}) + c_7 \deg X) D^{t+1-s}, \end{aligned}$$

with appropriate positive constants  $c_6, c_7$ . By Lemma 4.3,

$$-\widehat{\deg} (\bar{F}_Y(D)/E_{[n/2]}(D)) \leq 0,$$

and the claim follows with  $N = [e]$ .

If on the other hand  $\tilde{D} < 3\bar{D}$ , set  $l := [3\bar{D}(\tilde{D}) + 1]$ . Then, by Lemma 3.3

$$\log |f|^l \leq -c_1 \frac{h(\mathcal{X})}{\deg X} l\tilde{D} + \bar{c}_2 l\tilde{D}.$$

Replacing  $\tilde{D}$  by  $l\tilde{D} \geq 3\bar{D}$  in the above argumentation it follows in the same way that for  $N = 3$ ,  $D \geq l\tilde{D} \geq N\bar{D}$ , the inequality

$$H_{\mathcal{X}}(D) \geq (c_6 h(\mathcal{X}) - c_7 \deg X) D^{t+1-s}$$

holds.

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